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SEMIGROUP REPRESENTATIONS OF MEDIAL GROUPOIDS

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Abstract: For every medial groupoid G with $GG = G$, there exist a commutative monoid $S(+,0)$ and its two commuting automorphisms f and g such that $G \subseteq S$ and $xy = f(x) + g(y)$ for all $x,y \in G$.

Key words: Semigroup, representation, medial groupoid.

Classification: 08A05

In the past, a considerable attention was paid to the problem of representations of medial groupoids by means of commutative semigroups and their commuting endomorphisms (see e.g. [1],[2],[3],[6],[7],[8],[9] and [10]). In the present paper, we are going to show that all medial groupoids without irreducible elements and all commutative medial groupoids have semigroup representations.

1. Preliminaries. Throughout this paper, let E be a free monoid over a two-element set $\{\alpha, \beta\}$. Every element e of E can be written uniquely as $e = a_1 \dots a_n$ for some $n \geq 0$ and $a_1, \dots, a_n \in \{\alpha, \beta\}$. We put $d(e) = n$ and $E_m = \{f \in E; d(f) = m\}$ for all $m \geq 0$. Further, $l(e) = \mathrm{card}\{i; a_i = \alpha\}$, $r(e) = \mathrm{card}\{i; a_i = \beta\}$ and $E_{i,j} = \{e \in E; (l(e), r(e)) = (i,j)\}$.

Let X be a non-empty set and SW'_X the free algebra over X in the variety of universal algebras of the type $\{+, 0, \alpha, \beta\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$. Elements from SW'_X are called semiterms over X. Every semiterm can be expressed uniquely (up to the order of summands) in the form $s = e_1 x_1 + \dots + e_n x_n$, where $n \geq 0$, $e_i \in E$ and $x_i \in X$. We put $d(s) = \max d(e_i)$, $I^*(s) = \{e_1, \dots, e_n\}$, $I(s) = \{e \in E; ef \in I^*(s)$ for some $f \in E\}$.

Define a binary operation (denoted multiplicatively) on SW'_X by $rs = \alpha r + \beta s$. We obtain a groupoid SW. The subgroupoid W_X generated by X is an absolutely free groupoid over X and its elements are called terms.

Let $r \in W$ be a term and $e \in I(r)$. Then there exists a unique pair u, v such that u is a semiterm, v is a term and $r = u + ev$. We put $v = r_{[e]}$.

2. Linear representations of medial groupoids. Let X be a non-empty set. Denote by F'_X the free algebra over X in the variety $\mathcal{R}$ of universal algebras of the type $\{+, 0, \alpha, \beta\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$, $\beta\alpha x = \alpha\beta x$. Every element of F'_X can be written in the form $u = \sum_{i=1}^{k} \alpha^{n_i} \beta^{m_i} x_i$, where r, n_i, m_i are non-negative integers and $x_i \in X$; this expression is unique up to the order of summands. We put $d(u) = \max (n_i + m_i)$.

Define a multiplication on F'_X by $uv = \alpha u + \beta v$. The set F'_X together with this operation is a groupoid which will be denoted by F_X. Moreover, we can identify the set F_X with a subset of SW_X. It is easy to see that the groupoid F_X is a medial cancellation groupoid, and so it is entropic (recall that a groupoid is said to be medial if it satisfies the identity $xy.uv = xu.yv$ and it is said to be entropic if it is a homomorphic image of a medial cancellation groupoid). Denote by G_X the subgroupoid of F_X generated by X. According to [4, Theorem 2.1], G_X is a free entropic groupoid over X.

Let G be a groupoid. By a linear representation of G we mean an algebra $S(+,0,f,g) \in \mathcal{R}$ (i.e., $S(+,0)$ is a commutative monoid and f,g are commuting 0-preserving endomorphisms of $S(+)$) together with an element $e \in S$ such that G is a subset of S and $xy = f(x) + g(y) + e$ for all $x,y \in G$. The representation is called exact if $S = G$ and it is called convex if $e = 0$.

Using the fact that the underlying semigroups of free $\mathcal{R}$-algebras are cancellative, it is easy to show that groupoids with linear representations are entropic. Conversely, every medial groupoid containing an element a such that the corresponding translations are permutations has an exact linear representation (see [9]). Further, every regular medial division groupoid has an exact linear representation (see [6]) and every medial cancellation groupoid has a convex linear representation.

3. <u>Representations of medial groupoids without irreducible elements</u>. The purpose of this section is to prove the following

Theorem 1. Let G be a medial groupoid such that $GG = G$. Then G has a convex linear representation $S(+,0,f,g)$ such that f, g are automorphisms of $S(+)$.

The proof of this result will be divided into several lemmas.

Let $A(\circ)$ be a medial groupoid such that $A = A \circ A$. By [5, Proposition 4.3], $A(\circ)$ is entropic. In the following, we shall use the groupoids F_A and G_A defined in the preceding section. Denote by h the homomorphism of G_A onto $A(\circ)$ such that $h(x) = x$ for every $x \in A$. Further, for every $x \in A$, fix elements $p_\alpha(x)$, $p_\beta(x)$ in A such that $x = p_\alpha(x) \circ p_\beta(x)$. For every $x \in A$ and every $e \in E$, we define an element $p_e(x)$ of A by induction on $d(e)$ as follows: $p_1(x) = x$; $p_{e\alpha}(x) = p_\alpha(p_e(x))$; $p_{e\beta}(x) = p_\beta(p_e(x))$. Finally, for every $x \in A$ and every non-negative integer n, denote by $C(n,x)$ the element $\sum_{e \in E_n} \alpha^{l(e)} \beta^{r(e)} p_e(x)$ of F_A.

Lemma 2. Let $x \in A$ and $n \geq 0$. Then $C(n,x) \in G_A$ and $h(C(n,x)) = x$.

Proof. We shall proceed by induction on n. Let $n \geq 1$. Then $C(n,x) = C(n-1,p_\alpha(x)).C(n-1,p_\beta(x))$, and so $h(C(n,x)) = p_\alpha(x) \circ p_\beta(x) = x$.

For every element $u = \sum_{i=1}^{s} \alpha^{n_i} \beta^{m_i} x_i$ of H_A (see [4]) and every integer $n \geq d(u)$, let $D(n,u) = \sum_{i=1}^{s} \alpha^{n_i} \beta^{m_i} C(n-n_i-m_i, x_i)$.

Lemma 3. Let $u = \sum_{i=1}^{s} \alpha^{n_i} \beta^{m_i} x_i \in H_A$. Then:

(i) $D(n,u) \in G_A$ for every $n \geq d(u)$.

(ii) $D(n,u) = \sum_{i=1}^{s} \sum_{e \in E_{n-n_i-m_i}} \alpha^{n_i+l(e)} \beta^{m_i+r(e)} p_e(x_i)$. Moreover, $n_i + l(e) + m_i + r(e) = n$ for all $1 \le i \le s$ and $e \in E_{n-n_i-m_i}$.

Proof. For every $0 \le k \le n$ we have $c(n,k,D(n,u)) =$
$= \sum_{i=1}^{s} \sum_{e \in E_{n-n_i-m_i}} \binom{0}{k-n_i-l(e)} = \sum_{i=1}^{s} \operatorname{card}\{e \in E_{n-n_i-m_i};\ k =$
$= n_i + l(e)\} = \sum_{i=1}^{s} \binom{n-n_i-m_i}{k-n_i} = c(n,k,u) = \binom{n}{k}$, since $u \in H_A$ (see [4]). We have proved that $D(n,u) \in H_A$. By [4, Lemmas 2.10, 2.11], $D(n,u) \in G_A$ and the rest is clear.

Define a binary relation R on F_A as follows: $(u,v) \in R$ iff there are $w \in F_A$, $x,y,z \in A$ and $c,d \ge 0$ such that $u = w +$ $+ \alpha^c \beta^d x$, $v = w + \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z$ and $x = y \circ z$. Further, define a binary relation S on F_A by $(u,v) \in S$ iff there are $m \ge 0$ and $u_0, \dots, u_m \in F_A$ such that $u = u_0$, $v = u_m$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$ for all $1 \le i \le m$. Evidently, S is a congruence of the algebra $F_A' = F_A(+,0,\alpha,\beta)$.

Lemma 4. Let $(u,v) \in S$. Then $u \in H_A$ iff $v \in H_A$.

Proof. See [4, Lemma 2.9].

Lemma 5. Let $u,v \in H_A$ and $(u,v) \in R$. Then $h(D(n,u)) =$ $= h(D(n,v))$ for every $n \ge d(v)$.

Proof. Let $n \ge d(v)$. We have $u = w + \alpha^c \beta^d x$ and $v = w +$ $+ \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z$ for some $w \in F_A$, $c,d \ge 0$ and $x,y,z \in A$ with $x = y \circ z$. There is $w' \in F_A$ such that $D(n,u) = w' +$ $+ \alpha^c \beta^d C(n-c-d,x)$ and $D(n,v) = w' + \alpha^{c+1} \beta^d C(n-c-d-1,y) +$ $+ \alpha^c \beta^{d+1} C(n-c-d-1,z)$. We can express w' in the form $w' = \sum_{i=1}^{s} \alpha^{n_i} \beta^{m_i} x_i$; by Lemma 3, $D(n,u) \in H_A$ and $n_i + m_i = n$

for all $i = 1,\dots,s$. Hence, for every $0 \le k \le n$,

$\binom{n}{k} = c(n,k,w') + \sum_{e \in E_{n-c-d}} \binom{n-c-d-l(e)-r(e)}{k-c-l(e)} = \operatorname{card}\{i; n_i = k\} + \operatorname{card}\{e \in E_{n-c-d}; l(e) = k-c\} = \operatorname{card}\{i; n_i = k\} + \binom{n-c-d}{k-c}$, so that $\operatorname{card}\{i; n_i = k\} = \binom{n}{k} - \binom{n-c-d}{k-c}$. Now, let P designate the set of all $a_1 \dots a_n \in E_n$ such that $a_1 = \dots = a_c = \alpha$ and $a_{c+1} = \dots = a_{c+d} = \beta$. We have $\operatorname{card}(E_{k,n-k} \setminus P) = \binom{n}{k} - \binom{n-c-d}{k-c}$ for every $0 \le k \le n$. There exists a bijective mapping t_k of $E_{k,n-k} \setminus P$ onto $\{i; n_i = k\}$. Using this, let us define a mapping t of E_n into A as follows: $t(e) = x_{t_k(e)}$ for each $e \in E_{k,n-k} \setminus P$; $t(e) = p_{a_{c+d+1} \dots a_n}(x)$ for every $e = a_1 \dots a_n \in P$. It is now easy to check that $D(n,u) = \sum_{e \in E_n} \alpha^{l(e)} \beta^{r(e)} t(e)$.

There exists a unique term $r \in W_A$ with $I(r) = E_0 \cup \dots \cup E_n$ and $r_{[e]} = t(e)$ for every $e \in E_n$. Denote by g the homomorphism of W_A onto G_A such that $g(x) = x$ for all $x \in A$. We have $g(r) = D(n,u)$ by [4, Lemma 2.2]. Further, denote by f the element $a_1 \dots a_{c+d}$ of E_{c+d} with $a_1 = \dots = a_c = \alpha$ and $a_{c+1} = \dots = a_{c+d} = \beta$. Clearly, $g(r_{[f]}) = C(n-c-d,x)$. Moreover, there are elements $u_1,\dots,u_d,v_1,\dots,v_c \in W_A$ such that $r = ((((u_d(\dots(u_2(u_1.r_{[f]})))) v_1)v_2)\dots)v_c$. Consequently, $D(n,u) = g(r) = ((((u_d'(\dots(u_2'(u_1'.C(n-c-d,x)))))v_1')v_2')\dots)v_c'$ where $u_i' = g(u_i)$ and $v_i' = g(v_i)$ are elements from G_A. Put $q = ((((u_d'(\dots(u_2'(u_1'(C(n-c-d-1,y)C(n-c-d-1,z))))))v_1')\dots)v_c'$. Then $D(n,u) = \alpha^c \beta^d C(n-c-d,x) + \sum_{k=1}^{d} \alpha^{c+1} \beta^{d-k} u_k' + \sum_{j=1}^{c} \alpha^{c-1} \beta v_1'$ and so $w' = \sum_{k=1}^{d} \alpha^{c+1} \beta^{d-k} u_k' + \sum_{j=1}^{c} \alpha^{c-1} \beta v_1'$.

Further, $q = \alpha^{c}\beta^{d}(C(n-c-d-1,y).C(n-c-d-1,z)) +$
$+ \sum_{k=1}^{d} \alpha^{c+1}\beta^{d-k} u_k' + \sum_{j=1}^{c} \alpha^{c-1} \beta v_1' = \alpha^{c}\beta^{d}(C(n-c-d-1,y).$
$.C(n-c-d-1,z)) + w' = \alpha^{c+1}\beta^{d}C(n-c-d-1,y) +$
$+ \alpha^{c}\beta^{d+1}C(n-c-d-1,z) + w' = D(n,v)$. Therefore $h(D(n,u)) =$
$= ((((h(u_d')\circ(\dots\circ(h(u_1')\circ h(C(n-c-d,x)))))\circ h(v_1'))\circ h(v_2'))\circ$
$\circ \dots)\circ h(v_c')$ and $h(D(n,v)) =$
$= (((h(u_d')\circ(\dots\circ(h(u_1')\circ h(C(n-c-d-1,y)C(n-c-d-1,z)))))\circ$
$\circ\, h(v_1')\circ \dots)\circ h(v_c')$. But $h(C(n-c-d,x)) = x$, $h(C(n-c-d-1,y))=$
$= y$ and $h(C(n-c-d-1,z)) = z$ by Lemma 2. Finally, $x = y\circ z$ and we see that $h(D(n,u)) = h(D(n,v))$.

<u>Lemma 6</u>. Let $x,y\in A$ be such that $(x,y)\in S$. Then $x = y$.

Proof. There are elements $u_0,\dots,u_m\in F_A$ such that $x = u_0$, $y = u_m$ and $(u_{i-1},u_i)\in R\cup R^{-1}$. By Lemma 4, $u_i\in H_A$. Let n be such that $n\geq d(u_i)$ for all i. By 3.5, $h(D(n,u_0)) =\dots= h(D(n,u_m))$. Thus $x = h(D(n,u_0)) = h(D(n,u_m)) = y$.

<u>Lemma 7</u>. Let $u,v\in F_A$ be such that either $(\alpha u,\alpha v)\in S$ or $(\beta u,\beta v)\in S$. Then $(u,v)\in S$.

Proof. It is easy to see that if $(p,q)\in R\cup R^{-1}$ and $p = \alpha r$ for some r then $q = \alpha s$ for some s and $(r,s)\in R\cup R^{-1}$; similarly for β.

<u>Lemma 8</u>. The groupoid $A(\circ)$ has a convex linear representation such that f, g are injective.

Proof. It follows from the definition of S and from Lemma 6 that an algebra isomorphic to $F_A(+,0,\alpha,\beta)/S$ is a convex linear representation of $A(\circ)$. Let $S(+,0,f,g)$ be such an algebra. By Lemma 7, both f and g are injective and preserve the element 0.

Now, Theorem 1 is an easy consequence of Lemma 8.

4. Representations of medial groupoids with zero and without irreducible elements

Proposition 9. Let G be a medial groupoid such that $GG = G$. Suppose that G contains a zero element o (i.e., $xo = o = ox$). Then G has a convex linear representation $S(+,0,f,g)$ such that f, g are automorphisms of $S(+)$ and $x + o = o$ for all $x \in S$.

The proof of this result will be divided into six lemmas. Let $A(o)$ be a medial groupoid with $A \circ A = A$ and let o be a zero element of $A(o)$. We keep the notation of the preceding section; in the present case, we can assume that $p_e(o) = o$ for every $e \in E$.

Lemma 10. Let $u = \hat{\sum}_{i=1} \alpha^{n_i} \beta^{m_i} x_i \in H_A$ be such that $x_i = o$ for some $1 \leq i \leq s$. Then $h(D(n,u)) = o$ for every $n \geq d(u)$.

Proof. $D(n,u) = \hat{\sum}_{i=1} \alpha^{n_i} \beta^{m_i} C(n-n_i-m_i, x_i) = \sum_{i=1}^{2^n} \alpha^{c_i} \beta^{d_i} y_i$ for some $c_i, d_i \geq 0$ and $y_i \in A$ such that $c_i + d_i = n$ and o appears among the elements y_i. Further, there is a $t \in W_A$ such that $I(t) = E_0 \cup \dots \cup E_n$, o is contained in t and $h(t) = D(n,u)$. Denote by g the homomorphism of W_A onto G_A such that $g(x) = x$ for all $x \in A$. We have $h(D(n,u)) = hg(t) = o$.

Let I be the set of all $u = \hat{\sum}_{i=1} \alpha^{n_i} \beta^{m_i} x_i \in F_A$ such that $x_i = o$ for some i and define a binary relation Q on F_A as follows: $(u,v) \in Q$ iff either $(u,v) \in S$ or there exist elements $w, z \in I$ with $(u,w) \in S$ and $(v,z) \in S$.

Lemma 11. I is an ideal of $F_A(+)$ and Q is a congruence of $F_A(+,0,\alpha,\beta)$.

Proof. Obvious.

Lemma 12. Let $x \in A$, $x \neq o$, $u \in F_A$ and $(x,u) \in S$. Then $u \notin I$.

Proof. There are $m \geq 0$ and elements $u_0,\dots,u_m \in F_A$ such that $x = u_0$, $u = u_m$ and $(u_{i-1},u_i) \in R \cup R^{-1}$. Moreover, there is a positive integer n such that $n \geq d(u_i)$ for all i. Since $x \in H_A$, we have $u_i \in H_A$ and $h(D(n,u_{i-1})) = h(D(n,u_i))$ by Lemma 5. However, $h(D(n,u_0)) = x$, and hence $h(D(n,u)) = x$. By Lemma 10, $u \notin I$.

Lemma 13. Let $x,y \in A$ be such that $(x,y) \in Q$. Then $x = y$.

Proof. Suppose $x \neq y$. Then at least one of these elements is different from o; by Lemma 12 and the definition of Q, we get $(x,y) \in S$. Now, $x = y$ by Lemma 7, a contradiction.

Lemma 14. Let $u,v \in F_A$ be such that either $(\alpha u,\alpha v) \in Q$ or $(\beta u,\beta v) \in Q$. Then $(u,v) \in Q$.

Proof. Easy.

Lemma 15. The groupoid A(o) has a convex linear representation $S(+,0,f,g)$ such that f, g are injective, $x + o = o$ for all $x \in S$ and $f(o) = o = g(o)$.

Proof. An algebra isomorphic to $F_A(+,0,\alpha,\beta)/Q$ has the required properties.

Now, Proposition 9 is an easy consequence of Lemma 15.

5. Linear representations of commutative medial groupoids

Theorem 16. Let G be a commutative medial groupoid. Then G has a convex linear representation $S(+,0,f,g)$ such that

$f = g$ and f is an automorphism of $S(+)$.

The proof of the result will be divided into four lemmas.

Let $A(\circ)$ be a commutative medial groupoid. We denote by h the unique homomorphism of CG_A onto $A(\circ)$ such that $h(x) = x$ for each $x \in A$ (see [4, section 3]).

Define a binary relation R on CF_A as follows: $(u,v) \in R$ iff there are $w \in CF_A$, $x,y,z \in A$ and $c \geq 0$ such that $u = w + \alpha^c x$, $v = w + \alpha^{c+1} y + \alpha^{c+1} z$ and $x = y \circ z$. Further, define a binary relation S on CF_A by $(u,v) \in S$ iff there are $m \geq 0$ and $u_0, \dots, u_m \in CF_A$ such that $u_0 = u$, $u_m = v$ and $(u_{i-1}, u_i) \in R \cup R^{-1}$.

Lemma 17. (i) S is a congruence of $CF_A(+,0,\alpha)$.

(ii) If $u,v \in CF_A$ and $(\alpha u, \alpha v) \in S$ then $(u,v) \in S$.

(iii) If $(u,v) \in S$ then $u \in CG_A$ iff $v \in CG_A$.

(iv) If $x,y,z \in A$ and $x = y \circ z$ then $(x, yz) \in S$.

Proof. Easy (see [4, Lemma 3.4]).

Lemma 18. Let $(u,v) \in R$ and $u,v \in CG_A$. Then $h(u) = h(v)$.

Proof. We have $u = w + \alpha^c x$ and $v = w + \alpha^{c+1} y + \alpha^{c+1} z$, $w \in CF_A$, $c \geq 0$, $x,y,z \in A$, $x = y \circ z$. Denote by g the homomorphism of W_A onto G_A with $g(x) = x$ for every $x \in A$. Then $g(t) = u$ for some $t \in W_A$. By [4, Lemma 3.2], there is an $e = a_1 \dots a_c \in I(t)$ such that $t_{[e]} = x$. It is easy to see that $t = [t_{[e]}, a_c, t_c, \dots, a_1, t_1]$ for some $t_1, \dots, t_c \in W_A$ (see [5, Proposition 1.7]). Put $u_1 = g(t_1), \dots, u_c = g(t_c)$. Then $u_1, \dots, u_c \in CG_A$ and $u = g(t) = (((x u_c) u_{c-1}) \dots) u_1$. Consequently, $u = \alpha^c x + \alpha^c u_c + \alpha^{c-1} u_{c-1} + \dots + \alpha u_1$, $v = \alpha^{c+1} y + \alpha^{c+1} z + \alpha^c u_c + \alpha^{c-1} u_{c-1} + \dots + \alpha u_1 = (((yz.u_c) u_{c-1}) \dots) u_1$. From this, $h(u) = h(v)$.

Lemma 19. Let $x,y \in A$ and $(x,y) \in S$. Then $x = y$.

Proof. Use Lemma 17(ii) and Lemma 18.

Lemma 20. The groupoid $A(o)$ has a convex linear representation $S(+,0,f,f)$ such that f is injective.

Proof. Use Lemma 17 and Lemma 19.

Now, Theorem 16 is an easy consequence of Lemma 20.

6. Remarks

Proposition 21. Let f and g be commuting endomorphisms of a commutative semigroup $S(+)$ and $e \in S$. Put $xy = f(x) + g(y) + e$ for all $x,y \in S$ and suppose that the medial groupoid S is divisible. Then it is regular.

Proof. Let $L_a(b) = a + b$ for all $a,b \in S$. Denote by T the set of all $a \in S$ such that L_a is a projective transformation of S. Since the groupoid S is divisible, $e \in T$ and $f(a)$, $g(a) \in T$ for each $a \in S$. On the other hand, T is a subsemigroup and, moreover, an abelian group. The rest is clear.

Example 22. Let G be a non-regular medial division groupoid (see [7]). Then G has a convex linear representation. According to Proposition 21, G has no exact linear representation.

Example 23. Let $X = \{x,y\}$ be a two-element set. Denote by L the twelve-element subset of G_X formed by the elements x, xx, $xx.x$, $x.xx$, $xx.xx$, $(xx)(xx.x)$, y, xy, $xy.x$, $(xy.x)(xx)$, $(xy.x)(x.xx)$, $((xy.x)(xx))((xx)(xx.x))$. It is easy to see that $J = G_X \setminus L$ is an ideal of G_X. Put $r = (J \times J) \cup \mathrm{id}$ and $A = G_X/r$; it is clear that r is a congruence. We obtain thus an entropic groupoid A. On the other hand, it is easy to check that A has no convex linear representation.

Remark 24. The following problem seems to be open: Has every entropic groupoid a linear representation?

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