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NOTES ON GENERALIZED PRIME AND COPRIME MODULES I.

Josef JIRÁSKO

Abstract: The article continues the study of prime and coprime modules which were introduced by L. Bican, P. Jambor, T. Kepka, P. Němec in [7]. The concept of semiprime module which generalizes the notion of semiprime ideal as well as various generalizations of prime and semiprime modules are given. Numerous results known on prime (semiprime) rings and prime radical can be transferred to the modules. Rings in which every module is generalized prime are characterized. The preradical approach makes the dualization of these concepts possible; this leads to the definition of generalized coprime modules to which the second part of the article is dedicated.

Key words: Prime modules, semiprime modules, their generalizations, prime radical.

Classification: 16A12

In the following R stands for an associative ring with unit element and R-mod denotes the category of all unitary left R-modules.

A preradical r for R-mod is a subfunctor of the identity functor i.e. r assigns to every module M its submodule r(M) such that every homomorphism $f: M \longrightarrow N$ induces a homomorphism from r(M) into r(N) by restriction.

A module M is r-torsion if r(M) = M and r-torsionfree if r(M) = 0. The class of all r-torsion (r-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).

A preradical r is said to be

- idempotent if $r(M) \in \mathcal{T}_r$ for every module M,
- a radical if $M/r(M) \in \mathcal{F}_r$ for every module M,
- hereditary if for every module M and every monomorphism $f: A \longrightarrow r(M)$, $A \in \mathcal{T}_r$,
- pseudohereditary if for every projective module M and every monomorphism $f: A \longrightarrow r(M)$, $A \in \mathcal{T}_r$,
- superhereditary if it is hereditary and $\mathcal{T}_r$ is closed under direct products,
- cohereditary if for every module M and every epimorphism $f: M/r(M) \longrightarrow A$, $A \in \mathcal{F}_r$.

The idempotent core $\bar{r}$ of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r-torsion submodules K of M, the cohereditary core ch(r) by $ch(r)(M) = r(R)M$, $M \in$ $\in$ R-mod.

The superhereditary (cohereditary) preradical corresponding to a two-sided ideal I is defined by $s(M) = \{m \in M;\ Im = 0\}$ $(s(M) = IM)$, $M \in$ R-mod. The injective hull of M will be denoted by E(M).

A submodule N of a module M is characteristic in M, if there is a preradical r such that $N = r(M)$.

For a non-empty class of modules $\mathcal{A}$ $p^{\mathcal{A}}$ denotes the radical defined by $p^{\mathcal{A}}(M) = \bigcap \operatorname{Ker} f$, $f \in \operatorname{Hom}_R(M,A)$, $A \in \mathcal{A}$.

A module M is pseudo-injective if $p^{\{M\}}$ is hereditary. A module P is strongly M-projective if $P/(0:M)P$ is projective in $R/(0:M)$-mod.

A ring R is a left VS-ring if every module is pseudo-injective. A ring R is left quasi-hereditary if every two-sided ideal is projective as a left module.

Proposition 0.1. Let $M \in$ R-mod. Then the following are equivalent

(i) $p^{\{M\}}$ is pseudohereditary ($\overline{p^{\{M\}}}$ is pseudohereditary),

(ii) $ch(p^{\{M\}}) = ch(p^{\{E(M)\}})$ ($ch(\overline{p^{\{M\}}}) = ch(p^{\{E(M)\}})$),

(iii) $(0:M) = (0:E(M))$ ($\overline{p^{\{M\}}}(R) = (0:E(M))$),

(iv) $ch(p^{\{M\}})(E(M)) = 0$ ($ch(\overline{p^{\{M\}}})(E(M)) = 0$).

Proof. Obvious.

Finally Soc(J) will be denoted the Socle (Jacobson radical) and $\mathbb{N}$ the set of all natural numbers; zer denotes the zero functor.

§ 1. Prime and semiprime modules

1.1. A module $M \in$ R-mod is called

- prime if $p^{\{N\}}(M) = 0$ for every nonzero submodule N of M,
- pseudoprime if $ch(p^{\{N\}})(M) = 0$ for every nonzero submodule N of M,
- i-prime if $\overline{p^{\{N\}}}(M) = 0$ for every nonzero submodule N of M,
- i-pseudoprime if $ch(\overline{p^{\{N\}}})(M) = 0$ for every nonzero submodule N of M,
- semiprime if $N \cap p^{\{N\}}(M) = 0$ for every nonzero submodule N of M,
- pseudo-semiprime if $N \cap ch(p^{\{N\}})(M) = 0$ for every nonzero submodule N of M,
- i-semiprime if $N \cap \overline{p^{\{N\}}}(M) = 0$ for every nonzero submodule N of M,
- i-pseudo-semiprime if $N \cap ch(\overline{p^{\{N\}}})(M) = 0$ for every nonzero

submodule N of M.

For modules M, N and their submodules $A \subseteq M$ and $B \subseteq N$ let us define $t(A,M,B,N)$ by $t(A,M,B,N) = \sum f(A)$, $f \in \mathrm{Hom}_R(M.B)$.

Proposition 1.2. Let $M \in R$-mod and $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M. Then

(i) M is prime if and only if $p^{\{M\}} = p^{\{N\}}$ for every nonzero submodule N of M if and only if $t(A,M,B,M) \neq 0$ for all nonzero submodules $A,B \subseteq M$,

(ii) M is pseudoprime if and only if $ch(p^{\{M\}}) = ch(p^{\{N\}})$ for every nonzero submodule N of M if and only if $(0:M) = (0:N)$ for every nonzero submodule N of M if and only if $t(A,P,B,M) \neq 0$ for all $A \subseteq P$, $A \nsubseteq K$ and $0 \neq B \subseteq M$,

(iii) M is i-prime if and only if $\overline{p^{\{M\}}} = \overline{p^{\{N\}}}$ for every nonzero submodule N of M if and only if $t(A,A,B,M) \neq 0$ for all nonzero submodules $A,B \subseteq M$,

(iv) M is i-pseudoprime if and only if $t(A,A,B,M) \neq 0$ for all $A \subseteq P$, $A \nsubseteq K$ and $0 \neq B \subseteq M$,

(v) M is semiprime if and only if $t(A,M,A,M) \neq 0$ for every nonzero submodule A of M if and only if $t(A,M,B,M) \neq 0$ for all submodules $A,B \subseteq M$ with $A \cap B \neq 0$,

(vi) M is pseudo-semiprime if and only if $N \cap (0:N)M = 0$ for every nonzero submodule N of M if and only if $t(A,P,g(A),M) \neq 0$ for every $A \subseteq P$, $A \nsubseteq K$ if and only if $t(A,P,B,M) \neq 0$ for all $A \subseteq P$, $B \subseteq M$ with $g^{-1}(B) \cap A \nsubseteq K$,

(vii) M is i-semiprime if and only if $t(A,A,B,M) \neq 0$ for all submodules $A,B \subseteq M$ with $A \cap B \neq 0$,

(viii) M is i-pseudo-semiprime if and only if $t(A,A,B,M) \neq 0$ for all $A \subseteq P$, $B \subseteq M$ with $A \cap g^{-1}(B) \nsubseteq K$.

Proof. (i) was proved in [7].

(viii). Suppose M is i-pseudo-semiprime, $A \subseteq P$, $B \subseteq M$, $A \cap g^{-1}(B) \nsubseteq K$ and $t(A,A,B,M) = 0$. Then $A = \overline{p^{\{B\}}}(A)$ and hence $g(A) \subseteq ch(\overline{p^{\{B\}}})(M)$. Thus $g(A \cap g^{-1}(B)) = B \cap g(A) \subseteq B \cap ch(\overline{p^{\{B\}}})(M) =$ $= 0$, a contradiction.

On the contrary if $0 \neq N \subseteq M$ and $N \cap ch(\overline{p^{\{N\}}})(M) \neq 0$ then set $A =$ $= \overline{p^{\{N\}}}(P)$ and $B = N$. Then $t(A,A,B,M) \neq 0$ since $A \cap g^{-1}(B) \nsubseteq K$. Hence $A \nsubseteq \overline{p^{\{N\}}}(A) = A$, a contradiction.

The remaining assertions can be proved similarly.

Remark 1.3. In Proposition 1.2 N and B can be replaced by N cyclic and B cyclic.

Proposition 1.4. Let $M \in R$-mod.

If M is projective then

(i) M is prime if and only if M is pseudoprime,

(ii) M is i-prime if and only if M is i-pseudoprime,

(iii) M is semiprime if and only if M is pseudosemiprime,

(iv) M is i-semiprime if and only if M is i-pseudo-semiprime.

If M is uniform then

(v) M is prime if and only if M is semiprime,

(vi) M is i-prime if and only if M is i-semiprime,

(vii) M is pseudoprime if and only if M is pseudo-semiprime,

(viii) M is i-pseudoprime if and only if M is i-pseudo-semiprime.

Proof. Obvious.

Proposition 1.5. Every completely reducible module is semiprime.

Proof. Obvious.

Clearly, the classes of all prime, i-prime, pseudoprime, i-pseudoprime, semiprime, i-semiprime, pseudo-semiprime and i-pseudo-semiprime modules are closed under submodules.

<u>Proposition 1.6</u>. Let N be a submodule of M and $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M. Then

(i) M/N is pseudoprime if and only if $t(A,P,B,M) \nsubseteq N$ whenever $g(A) \nsubseteq N$ and $B \nsubseteq N$,

(ii) M/N is pseudo-semiprime if and only if $t(A,P,B,M) \nsubseteq$ $\nsubseteq N$ whenever $g(A) \cap B \nsubseteq N$;

if M is projective then

(iii) M/N is prime implies $t(A,M,B,M) \nsubseteq N$ whenever $A \nsubseteq N$ and $B \nsubseteq N$,

(iv) M/N is semiprime implies $t(A,M,B,M) \nsubseteq N$ whenever $A \cap B \nsubseteq N$;

if N is a characteristic submodule of M then

(v) if $t(A,M,B,M) \nsubseteq N$ whenever $A \nsubseteq N$ and $B \nsubseteq N$ then M/N is prime,

(vi) if $t(A,M,A,M) \nsubseteq N$ whenever $A \nsubseteq N$ then M/N is semiprime.

<u>Proof</u>. (iii) and (v) were proved in [7].

(ii). Suppose M/N is pseudo-semiprime $A \subseteq P$, $B \subseteq M$ and $g(A) \cap$ $\cap B \nsubseteq N$. Then $t(A,P,(B+N)/N,M/N) \neq 0$ since $(g(A)+N)/N \cap (B+N)/N \neq$ $\neq 0$ and M/N is pseudo-semiprime. Thus $f(A) \neq 0$ for some $f:P \longrightarrow$ $\longrightarrow (B+N)/N$. Therefore there is a homomorphism $h:P \longrightarrow B$ such that $\pi \circ h = f$, where π is the natural epimorphism. Thus $h(A) \nsubseteq N$ and consequently $t(A,P,B,M) \nsubseteq N$.

On the other hand if $A \subseteq P$, $B/N \subseteq M/N$ such that $(g(A)+N)/N \cap$ $\cap B/N \neq 0$ then $t(A,P,B,M) \nsubseteq N$ since $g(A) \cap B \nsubseteq N$. Hence there is a homomorphism $f:P \longrightarrow B$ with $f(A) \nsubseteq N$. Thus $\bar{f} = \pi \circ f:P \longrightarrow B/N$, where π is the natural epimorphism, $\bar{f}(A) \neq 0$ and consequently M/N is pseudo-semiprime since $t(A,P,B/N,M/N) \neq 0$.

The remaining assertions can be proved similarly.

(Left) ideals I with the property R/I to be prime (pseudoprime) were described in [7].

Proposition 1.7. The following are equivalent for a left ideal I in R:

(i) R/I is semiprime,

(ii) for every $x \in R \setminus I$ there is $y \in I+Rx$ with $Iy \subseteq I$ and $xy \notin I$,

(iii) for every $x \in R \setminus I$ there is $z \in R$ with $Izx \subseteq I$ and $xzx \notin I$.

Proof. The equivalence of (ii) and (iii) is cbvious.

(i) implies (ii). If $x \in R \setminus I$ then there is a homomorphism $f:R/I \longrightarrow (I+Rx)/I$ with $f(x+I) \neq 0$. Set $f(1+I) = y+I$. Then $Iy \subseteq I$ and $xy \notin I$.

(ii) implies (i). If $I \subsetneqq K \subseteq R$ and $x \in K \setminus I$ then there is $y \in I+Rx \subseteq K$ with $Iy \subseteq I$ and $xy \notin I$. Let us define a homomorphism $f:R/I \longrightarrow K/I$ by $f(r+I) = ry+I$. Then $f(x+I) \neq 0$.

Proposition 1.8. The following are equivalent for a left ideal I in R:

(i) R/I is pseudo-semiprime,

(ii) if A, B are left ideals then $A \cdot B \subseteq I$ implies $A \cap B \subseteq I$,

(iii) if A is a twosided ideal and B is a left ideal with $A \cdot B \subseteq I$ then $A \cap B \subseteq I$,

(iv) if A is a left ideal then $A^2 \subseteq I$ implies $A \subseteq I$,

(v) if $a \in R$, $aRa \subseteq I$ then $a \in I$.

__Proof__. (i) is equivalent to (ii). It follows immediately from Proposition 1.6 (ii). The rest is clear.

__Corollary 1.9__. The following are equivalent for a twosided ideal I in R:

(i) R/I is semiprime,

(ii) R/I is pseudo-semiprime,

(iii) if A, B are twosided ideals then $A \cdot B \subseteq I$ implies $A \cap B \subseteq I$,

(iv) if A is a twosided ideal then $A^2 \subseteq I$ implies $A \subseteq I$.

__Proof__: It follows immediately from Propositions 1.6 and 1.8.

__Remark__: J. Dauns showed that if M is pseudoprime and N is a complement in M then M/N is pseudoprime ([8], Prop. 2.7).

__Proposition 1.10__. If I is a twosided ideal in R and s is the superhereditary preradical corresponding to I then

(i) M is pseudoprime implies $s(M) = M$ if $s(M) \neq 0$,

(ii) M is pseudo-semiprime implies $s(M) \cap IM = 0$.

Moreover if I is idempotent then

(iii) M is i-pseudoprime implies $s(M) = M$ if $s(M) \neq 0$,

(iv) M is i-pseudo-semiprime implies $s(M) \cap IM = 0$.

__Proof.__ (iii). Let $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M. As it is easy to see $IP \subseteq p^{\{s(M)\}}(P)$ and hence $IP \subseteq \overline{p^{\{s(M)\}}}(P)$ since I is idempotent. Now $IM = g(IP) \subseteq g(\overline{p^{\{s(M)\}}}(P)) = ch(\overline{p^{\{s(M)\}}})(M)$ and consequently $IM = 0$ if $s(M) \neq 0$ and M is i-pseudo-prime.

The rest can be proved similarly as above.

The following lemma has a technical character. We present it here without the proof.

Lemma 1.11. Let $M \in R$-mod and $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M. For $A \subseteq P$, $B \subseteq M$ let us denote $T(A,P,B,M) = \sum f(A)$, $f \in \mathrm{Hom}_R(P,P)$ and $\mathrm{Im}\, g \circ f \subseteq B$. Then

(i) $g(T(A,P,B,M)) = t(A,P,B,M)$ for all $A \subseteq P$ and $B \subseteq M$,

(ii) $t(T(A,P,B,M),P,C,M) \subseteq t(A,P,t(g^{-1}(B),P,C,M),M)$ for all $A \subseteq P$ and $B,C \subseteq M$,

(iii) $t(t(A,P,B,M),M,C,M) = t(A,P,t(B,M,C,M),M)$ for all $A \subseteq P$ and $B,C \subseteq M$.

Proposition 1.12. Let N be a submodule of M and C be the largest characteristic submodule of M contained in N. Then

(i) if M/N is pseudoprime then M/C is so,

(ii) if M/N is pseudo-semiprime then M/C is so.

Moreover if M is projective then

(iii) if M/N is pseudoprime then M/C is prime,

(iv) if M/N is pseudo-semiprime then M/C is semiprime.

Proof. (iii) and (iv). It follows from (i) and (ii) and Proposition 1.6.

(ii). Let $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M, $A \subseteq P$ and $B \subseteq M$ such that $g(A) \cap B \nsubseteq C$. Suppose $t(A,P,B,M) \subseteq C$. Then $t(T(A,P,M,M),P,t(B,M,M,M),M) \subseteq$ $\subseteq t(A,P,t(P,P,t(B,M,M,M),M),M) \subseteq t(A,P,t(B,M,M,M),M) =$ $= t(t(A,P,B,M),M,M,M) \subseteq t(C,M,M,M) \subseteq C \subseteq N$ by Lemma 1.11 (ii) and (iii) since C is characteristic in M. Now M/N is pseudo-semiprime, hence $X = t(A,P,M,M) \cap t(B,M,M,M) = g(T(A,P,M,M)) \cap$ $\cap t(B,M,M,M) \subseteq N$. Further, $t(A,P,M,M)$ and $t(B,M,M,M)$ are characteristic in M, hence X is characteristic in M. Thus $X \subseteq C$ and therefore $g(A) \cap B \subseteq X \subseteq C$, a contradiction.

(i). It can be proved similarly as in (ii).

Corollary 1.13. (i) If M is pseudoprime then $P/(0:M)P$ is prime for every projective module P.

(ii) If M is pseudo-semiprime then $P/(0:M)P$ is semiprime for every projective module P.

Proof. (ii). Let $0 \longrightarrow K \hookrightarrow F \longrightarrow M \longrightarrow 0$ be a free presentation of M. As it is easy to see $(0:M)F$ is the largest characteristic submodule of F contained in K and $F/(0:M)F$ is semiprime by Proposition 1.12. Hence $R/(0:M)$ is semiprime and one may check easily that $P/(0:M)P$ is semiprime for every projective module P.

(i). It can be made similarly as in (ii).

Corollary 1.14. Consider the following conditions:

(i) M is prime (semiprime),

(ii) M is pseudoprime (pseudo-semiprime),

(iii) $R/(0:M)$ is a prime (semiprime) ring,

(iv) $R/(0:M)$ is a prime (semiprime) R-module,

(v) there is a prime (semiprime) module N with $(0:N) = (0:M)$,

(vi) every submodule Q with $(0:M)P \cap Q = 0$ of a strongly M-projective module P is prime (semiprime).

Then the conditions (iii),(iv),(v) and (vi) are equivalent, (i) implies (ii) and (ii) implies (iii). Moreover if $M = Ra$, where $(0:a)$ is a twosided ideal then all conditions are equivalent.

Proof. It follows immediately from Corollaries 1.13 and 1.9.

Corollary 1.15. Let M be a module without nontrivial characteristic submodules. If $J(M) \neq M$ then M is pseudoprime.

Proof. It follows from Proposition 1.12.

Proposition 1.16. Let $M \in R$-mod. Then

(i) if M is prime and $\mathrm{Soc}(M) \neq 0$ then $J(M) = 0$,

(ii) if M is pseudoprime and $\mathrm{Soc}(M) \neq 0$ then $J(R)M = 0$,

(iii) if M is i-prime and $\mathrm{Soc}(M) \neq 0$ then $\bar{J}(M) = 0$,

(iv) if M is i-pseudo-prime and $\mathrm{Soc}(M) \neq 0$ then $\bar{J}(R)M = 0$,

(v) if M is semiprime then $\mathrm{Soc}(M) \cap J(M) = 0$,

(vi) if M is pseudo-semiprime then $\mathrm{Soc}(M) \cap J(R)M = 0$,

(vii) if M is i-semiprime then $\mathrm{Soc}(M) \cap \bar{J}(M) = 0$,

(viii) if M is i-pseudo-semiprime then $\mathrm{Soc}(M) \cap \bar{J}(R)M = 0$.

Proof. Obvious.

Proposition 1.17. (i) Every module is prime if and only if every nonzero module is a cogenerator if and only if R is isomorphic to a matrix ring over a skew-field.

(ii) Every module is pseudo-prime if and only if every nonzero module is faithful if and only if R is a simple ring.

(iii) Every module is semiprime if and only if R is a left VS-ring if and only if every radical is hereditary.

(iv) Every module is pseudo-semiprime if and only if $(0:M) = (0:E(M))$ for every module M if and only if every radical is pseudohereditary if and only if every left ideal is idempotent.

(v) Every module is i-prime if and only if $\overline{p^{\{M\}}} = zer$ for every nonzero module M if and only if R is isomorphic to a matrix ring over local left and right perfect ring.

(vi) Every module is i-pseudoprime if and only if $\overline{p^{\{M\}}}(R) = 0$ for every nonzero module M. Moreover if $\mathrm{Soc}(R) \neq 0$ it is equivalent to: R is isomorphic to a matrix ring over local right perfect ring and $\overline{ch(J)}(R) = 0$; if R is left quasi-hereditary it is equivalent to: R is a ring without nontrivial idempotent two-sided ideals.

(vii) Every module is i-semiprime if and only if $\overline{p^{\{M\}}}$ is hereditary for every module M if and only if every idempotent radical is hereditary.

(vii) Every module is i-pseudo-semiprime if and only if $\overline{p^{\{M\}}}$ is pseudohereditary for every module M if and only if every idempotent radical is pseudohereditary.

<u>Proof</u>. The equivalence of the first and the last condition of (i) was proved in [7]. Further every module is prime if and only if $p^{\{M\}}$ = zer for every nonzero module M iff R has no nontrivial radicals.

(ii). Every module is pseudo-prime iff $ch(p^{\{M\}})$ = zer for every nonzero module M iff R has no nontrivial cohereditary radicals.

The rest is clear.

For (iii) see [13], Proposition 3.1. The rest is clear.

(vii) and (viii) can be made similarly as in (iii).

(iv). It follows from Proposition 0.1 and the fact that every radical is pseudohereditary iff every cohereditary radical is hereditary. The rest follows from [15], Proposition VI.1.29.

(v). As it is easy to see every module is i-prime iff $\overline{p^{\{M\}}}$= zer for every nonzero module M iff R has no nontrivial idempotent radicals.

The rest follows from [15], Proposition VI.1.24.

(vi). Every module is i-pseudoprime iff $ch(\overline{p^{\{M\}}})$ = zer if $M \neq 0$ iff either r(R) = R or r(R) = 0 for every idempotent radical and it suffices to use 15 , Proposition VI.1.23.

Let $\mathcal{A}$ ($\mathcal{B}$) be the class of all prime (pseudo-prime) modules. The prime radical $\mathcal{P}_{\mathcal{B}}$ (pseudo-prime radical $\mathcal{P}_1$) is defined as follows: $\mathcal{P} = p^{\mathcal{A}}$ ($\mathcal{P}_1 = p^{\mathcal{B}}$).

Proposition 1.18. Let $M \in$ R-mod and $0 \longrightarrow K \hookrightarrow P \xrightarrow{g} M \longrightarrow 0$ be a projective presentation of M. Then

(i) $\mathcal{P}(M) = \mathcal{P}_1(M)$ if M is projective,

(ii) $\mathcal{P}_1(M)$ is the set of all elements m of M with the following property: "whenever $\{m_i, i \in \mathbb{N}\} \subseteq M$, $\{b_i, i \in \mathbb{N}\} \subseteq P$ such that $m_1 = m$, $g(b_i) = m_i$, $b_{i+1} \in Rb_i$ and $m_{i+1} \in t(Rb_i, P, Rm_i, M)$ for all $i \in \mathbb{N}$ then there is $k \in \mathbb{N}$ with $m_k = 0$ ", provided that $K = (0:M)P$,

(iii) $\mathcal{P}_1(M) = p^{\mathcal{M}}(M)$ where $\mathcal{M}$ is the class of all pseudo-semiprime modules, provided that $K = (0:M)P$,

(iv) M is pseudo-semiprime if and only if $\mathcal{P}_1(M) = 0$, provided that $K = (0:M)P$,

(v) if M is projective then $\mathcal{P}(M)$ is the set of all elements m of M with the following property: whenever $\{m_i, i \in \mathbb{N}\} \subseteq$ $\subseteq M$ such that $m_1 = m$ and $m_{i+1} \in t(Rm_i, M, Rm_i, M)$ for all $i \in \mathbb{N}$ then there is $k \in \mathbb{N}$ with $m_k = 0$,

(vi) $\mathcal{P}(R) = p^{\mathcal{N}}(R)$, where $\mathcal{N}$ is the class of all semiprime modules,

(vii) if M is projective then M is semiprime if and only if $\mathcal{P}(M) = 0$,

(viii) if M is projective and for every submodule N of M N^k is defined inductively as follows: $N^1 = N$, $N^{k+1} =$ $= t(N^k, M, N, M) = t(N, M, N^k, M)$ for $k \in \mathbb{N}$ then M is semiprime if and only if M has no nonzero nilpotent submodules i.e. whenever $A \subseteq M$ and $A^k = 0$ for some $k \in \mathbb{N}$ then $A = 0$.

Proof. (i). It follows immediately from Proposition 1.12 (iii).

(ii) and (iii). Let X be the set of all elements m of M with the property which is given in (ii) and M/N be a pseudo-semiprime module. Suppose $X \nsubseteq N$. Then there are $x_1 \in X$, $x_1 \notin N$, $b_1 \in P$, $g(b_1) = x_1$. Suppose $\{x_1, \dots, x_k\} \subseteq X \setminus N$, $\{b_1, \dots, b_k\} \subseteq P$, $g(b_i) = x_i$, $b_i \in Rb_{i-1}$ and $x_i \in t(Rb_{i-1}, P, Rx_{i-1}, M)$ for $i \in \{2, \dots$ $\dots, k\}$. Then $t(Rb_k, P, Rx_k, M) \nsubseteq N$ since M/N is pseudo-semiprime and $Rx_k \nsubseteq N$. Hence there is $x_{k+1} \in t(Rb_k, P, Rx_k, M)$, $x_{k+1} \notin N$. Thus $x_{k+1} = ax_k$ for some $a \in R$ and it suffices to set $b_{k+1} = ab_k$. Hence $x_1 \notin X$, a contradiction. Therefore $X \subseteq p^{\mathcal{M}}(M)$, where $\mathcal{M}$ is the class of all pseudo-semiprime modules.

On the other hand if $x \in \mathcal{P}_1(M)$, $x \notin X$ then there are $\{m_i, i \in$ $\in \mathbb{N}\} \subseteq M$, $\{b_i, i \in \mathbb{N}\} \subseteq P$ such that $x = m_1$, $g(b_i) = m_i \neq 0$, $b_{i+1} \in Rb_i$ and $m_{i+1} \in t(Rb_i, P, Rm_i, M)$ for every $i \in \mathbb{N}$. Put $S =$ $= \{b_i, i \in \mathbb{N}\}$. Let C be a submodule of P maximal in the set of all submodules D of P with $D \supseteq K$, $D \cap S = \emptyset$ and D characteristic in P. Suppose that $A \subseteq P$, $B \subseteq M$ such that $g(A) \nsubseteq g(C)$, $B \nsubseteq g(C)$ and $t(A, P, B, M) \subseteq g(C)$. As it is easy to see $C \subsetneqq C + A \subseteq C + T(A, P, M, M)$, $C \subsetneqq C + g^{-1}(B) \subseteq C + g^{-1}(t(B, M, M, M))$ and $C + T(A, P, M, M)$, $C +$ $+ g^{-1}(t(B, M, M, M))$ are characteristic in P. Hence there is $k \in \mathbb{N}$ such that $Rb_k \subseteq (T(A, P, M, M) + C) \cap (g^{-1}(t(B, M, M, M)) + C)$. Thus $m_{k+1} \in t(Rb_k, P, Rm_k, M) \subseteq t((T(A, P, M, M) + C), P, (t(B, M, M, M) +$ $+ g(C)), M) \subseteq t(T(A, P, M, M), P, t(B, M, M, M), M) + t(C, P, t(B, M, M, M), M) +$ $+ t(T(A, P, M, M), P, g(C), M) + t(C, P, g(C), M) \subseteq t(A, P, t(P, P, t(B, M, M, M,$ $M), M) + t(C, P, M, M) + g(C) \subseteq t(A, P, t(B, M, M, M), M) + g(C) =$ $= t(t(A, P, B, M), M, M, M) + g(C) \subseteq t(g(C), M, M, M) + g(C) \subseteq g(C)$ by Lemma 1.11 (ii) and (iii). Thus $m_{k+1} \in g(C) \cap g(S) = \emptyset$, a contradiction. Therefore $M/g(C)$ is pseudoprime and $x \in g(C) \cap \cap g(S) = \emptyset$, a contradiction. Thus $\mathcal{P}_1(M) \subseteq X$.

(iv). If M is pseudo-semiprime then $\mathcal{P}_1(M) = 0$ by (iii). Conversely if $\mathcal{P}_1(M) = 0$, $A \subseteq P$, $A \nsubseteq K$ then $g(A) \nsubseteq \mathcal{P}_1(M)$. Hence there is a homomorphism $f: M \longrightarrow N$, where N is pseudoprime and $f(g(A)) \neq 0$. Let $0 \longrightarrow K_1 \hookrightarrow P_1 \xrightarrow{g_1} N \longrightarrow 0$ be a projective presentation of N. Then there is a homomorphism $h: P \longrightarrow P_1$ with $g_1 \circ h = f \circ g$. Further $k(h(A)) \neq 0$ for some homomorphism $k: P_1 \longrightarrow \longrightarrow f(g(A))$ since N is pseudoprime. Now $f \circ p = k \circ h$ for some homomorphism $p: P \longrightarrow g(A)$ and hence $t(A,P,g(A),M) \neq 0$. Thus M is pseudo-semiprime by Proposition 1.2 (vi).

(v) follows immediately from (ii) and (i).

(vi) Follows immediately from (i),(iii) and Proposition 1.12 (iv).

(vii) Follows from (i) and (iv).

(viii). Obvious (see Lemma 1.11 (iii)).

<u>Proposition 1.19</u>. The following conditions are equivalent:

(i) every pseudo-semiprime module is completely reducible,

(ii) $R/\mathcal{P}(R)$ is a completely reducible ring.

Proof. (i) implies (ii). $\mathcal{P}_1(R/\mathcal{P}(R)) = 0$ hence $R/\mathcal{P}(R)$ is pseudo-semiprime and $R/\mathcal{P}(R)$ is completely reducible by assumption.

(ii) implies (i). If M is pseudo-semiprime then $\mathcal{P}_1(R)M = p^{\mathcal{M}}(R)M \subseteq p^{\mathcal{M}}(M) = 0$ where $\mathcal{M}$ is the class of all pseudo-semiprime modules by Proposition 1.18 and consequently M is completely reducible.

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