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REPRESENTING GRAPHS BY MEANS OF STRONG AND WEAK PRODUCTS

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Abstract: Representability of graphs by means of product-like constructions from simpler ones is studied. An estimate of dimension of the strong product of two graphs is presented.

Key words: Product, weak product, strong product, dimension.

Classification: 05C99

The aim of this paper is to discuss some aspects of representing graphs as induced subgraphs of results of product-like operations carried out with simpler graphs. There are three such operations one usually encounters (for description see 1.1): the (categorial) product, the weak product and the strong product. (In fact, there are categorial reasons why exactly these three kinds of products are of importance.) The representation by means of the (categorial) product and the resulting dimension characteristics have been studied recently in some intensity (e.g. [2],[7],[8],[6],[1], survey in [3]). Here we shall be concerned mostly with the other two types of products. In § 1 we will show that the weak product cannot be used as a tool for generating graphs, not even the bipartite graphs, from simpler ones. In §§ 2 and 3, the representation by means of strong powers of the path of length two is investigated and in § 4 a close connection of this and the categorial product representation of bipartite graphs is shown. For general graphs no similar connections hold as follows from remarks in § 5, the main aim of which, however, is to present an estimate of dimension of strong product.

<u>Conventions and notation:</u> A <u>graph</u> is a finite undirected graph without loops with the set of vertices $V(G)$ and the set of edges $E(G)$. Its cardinality, denoted by $|G|$, is the cardinality of $V(G)$.

We say that G is (or can be) <u>embedded</u> into H if it is isomorphic to an induced subgraph of H. A particular isomorphism of G with an induced subgraph of another graph is sometimes referred to as a <u>representation</u> of G.

Vectors $(x_1,\dots,x_n)$ will be often written as words $x_1x_2\dots$ $\dots x_n$, the concatenation of words is denoted by juxtaposition. A natural number n is viewed as the set of all smaller ones (thus, e.g., $2 = \{0,1\}$), but n-dimensional vectors are, as a rule, indexed from 1 to n rather than from 0 to n-1.

The word obtained by repeating i n-times will be denoted by

$\tilde{i}(n)$ (or simply $\tilde{i}$, if n is obvious).

The upper integral approximation of $\log_2 x$ is denoted by $\log^+ x$.

<u>Special graphs</u>: K_n is the complete graph with n vertices; $K(G)$ is the complete graph with the same set of vertices as G. D_n is the n-point discrete graph. P_n is the path $(n+1,\{\{i,i+1\} \mid i = 0,\dots,n-1\})$, C_n is the cycle $(n,\{\{i,i+1\} \mid i = 0,\dots,n-2\} \cup \{\{0,n-1\}\})$. In case of complex indices we

write P(n) instead of P_n.

1. Three products and representations using them

1.1. Let G_i, $i = 1,\dots,n$, be graphs. Graphs $\mathop{\times}_{i=1}^{n} G_i$, $\mathop{\Box}_{i=1}^{n} G_i$ and $\mathop{\boxtimes}_{i=1}^{n} G_i$ are defined as follows:

$V(\times G_i) = V(\Box G_i) = V(\boxtimes G_i) = \times V(G_i)$,

$\{(x_1,\dots,x_n),(y_1,\dots,y_n)\} \in E(\times G_i)$ iff $\forall i\ \{x_i,y_i\} \in E(G_i)$

$\{(x_1,\dots,x_n),(y_1,\dots,y_n)\} \in E(\Box G_i)$ iff $\exists j\ (\{x_j,y_j\} \in E(G_j)\ \&\ (i \neq j \Rightarrow x_i = y_i))$

$\{(x_1,\dots,x_n),(y_1,\dots,y_n)\} \in E(\boxtimes G_i)$ iff $\exists j\ (\{x_j,y_j\} \in E(G_j))\ \&\ \forall i\ (x_i \neq y_i \Rightarrow \{x_i,y_i\} \in E(G_i))$

We write $G_1 \times G_2$, ($G_1 \Box G_2$, $G_1 \boxtimes G_2$, resp.) for $\mathop{\times}_{i=1}^{2} G_i$ ($\mathop{\Box}_{i=1}^{2} G_i$, $\mathop{\boxtimes}_{i=1}^{2} G_i$, resp.) (one sees easily that $\times$, $\Box$ and $\boxtimes$ are associative - up to the "associativity" of cartesian product of sets - and that $\times G_i = G_1 \times G_2 \times \dots \times G_n$, etc.).

$\mathop{\times}_{i=1}^{n} G_i$ is simply usually referred simply as <u>product</u> of the graphs G_i, $\mathop{\Box}_{i=1}^{n} G_i$ as their <u>weak</u> (or <u>cartesian</u>) product, and $\mathop{\boxtimes}_{i=1}^{n} G_i$ as their <u>strong product</u>.

If $G_i = G$ for all i, $\mathop{\times}_{i=1}^{n} G_i$, $\mathop{\Box}_{i=1}^{n} G_i$ and $\mathop{\boxtimes}_{i=1}^{n} G_i$ are denoted (in this order) by

$$G^n,\ G^{\Box n},\ G^{\boxtimes n}.$$

1.2. <u>Information</u>: Every graph G can be represented as an induced subgraph of some K_k^n, the minimum necessary n is called dimension of G and denoted by

dim G (see e.g. [5],[2],[4])

Every bipartite graph G can be represented as an induced subgraph of P_3^n (see e.g. [9]), the minimum such n is called bipartite dimension of G and denoted by

$$\text{bid } G \text{ (see [8]).}$$

Every graph G can be represented as an induced subgraph of $P_2^{\boxtimes n}$ (this follows e.g. from [10; 4.6]; it will be obvious from the proof of 2.1(a) below). The minimum necessary n will be denoted by

$$\sigma(G).$$

(This, in essence, coincides with one of the dimension characteristics of tolerance spaces introduced in [11].)

In contrast with these facts, the weak product is a very weak tool for representing graphs. In fact, as we will show below in 1.3, a system of graphs $\mathcal{G}$ such that it generates all graphs by means of weak products and induced subgraphs does it without the products as well (i.e., for every G there is then an $H \in \mathcal{G}$ such that G is its induced subgraph). This follows very easily from the behavior of triangles in weak products (we are indebted to J. Nešetřil for this observation; meanwhile, we have been informed that this author has proved analogous results for various classes of graphs using Ramsey theory). The problem naturally arose whether after avoiding triangles the representation abilities improve. They do not, as will be seen in 1.5: the statement on $\mathcal{G}$ above holds even if we wish to generate just the bipartite graphs.

1.3. <u>Proposition</u>: For every graph G there is a graph H such that

(1) G is an induced subgraph of H,

(2) $|H| = |G| + 2$, and

(3) if H is embeddable into $\mathop{\square}\limits_{i=1}^{m} G_i$, it is embeddable into some of the G_i.

Proof. First, observe that if a triangle K_3 is embedded into $\mathop{\square}_{i=1}^{m} G_i$, say as $x_1 \dots x_n$, $y_1 \dots y_n$, $z_1 \dots z_n$ and if j is the coordinate such that, for $i \neq j$, $x_i = y_i$, then also $z_i = x_i = y_i$ for $i \neq j$. (Really, if $z_k \neq x_k$ for some $k \neq j$, we have $z_j = x_j$ and hence $z_j \neq y_j$ and $z_k \neq y_k$, so that z is not joined with y.)

Construct H as follows:

$V(H) = V(G) \cup \{a,b\}$ where $a,b \notin V(G)$, $a \neq b$,

$E(G) = E(G) \cup \{\{a,b\}\} \cup \{\{a,x\} \mid x \in V(G)\} \cup \{\{b,x\} \mid x \in V(G)\}$.

If H is embedded into $\square G_i$, consider the images of the triangles $\{x,a,b\}$.

1.4. Lemma. Let $D = (\{a,b,c,x,y\}, \{a,x\},\{a,y\},\{b,x\},\{b,y\},\{c,x\},\{x,y\}\})$ (see Fig.) be an induced subgraph of $\mathop{\square}_{i=1}^{n} G_i$. Then there is an r such that $a_i = b_i = c_i = x_i = y_i$ for $i \neq r$.

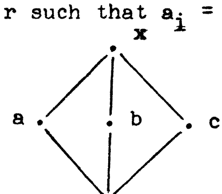

Fig.

Proof. The points x and y may differ either in two coordinates or in one. In the first case we have necessarily $a_i = b_i = c_i = x_i = y_i$ for $i \neq r,s$ and for z any of a,b,c one has either $z_r = x_r$ and $z_s = y_s$ or $z_r = y_r$ and $z_s = x_s$. But these are two possibilities and the vertices a,b,c are three. Thus, x and y differ in one coordinate $x_r \neq y_r$. Then x_r cannot be joined with y_r in G_r (x with y is not in D). Consequently necessarily $z_r \neq x_r, y_r$ for z = a,b,c which immediately yields $z_i = x_i = y$ for z = a,b,c, $i \neq r$.

1.5. Proposition. For every bipartite graph G there is a bipartite graph H such that

(1) G is an induced subgraph of H,

(2) $|H| = |G| + 4$, and

(3) If H is embeddable into $\square_{i=1}^{m} G_i$, it is embeddable into some of the G_i.

Proof. Consider a bipartition (P,Q) of G such that P contains all the isolated points. Take four distinct points u,a,d, $c \in P \cup Q$ and construct an H with bipartition $(P \cup \{u\}, Q \cup \{a,b,c\})$ by putting

$E(H) = E(G) \cup \{\{u,x\} \mid x \in Q \cup \{a,b,c\}\} \cup \{\{x,z\} \mid x \in P,\ z = a,b,c\}$.

Now observe that any point of V(H) lies in an induced subgraph of H isomorphic to D from 1.4 and that the coordinate r in 1.4 is uniquely determined by any two vertices of D.

§ 2. Estimates and values of δ for some graphs

2.1. Proposition. (a) $\delta(G) \leq |G|$,

(b) for each n there is a G with $|G| = 2n$ and $\delta(G) \geq n$.

Proof. (a) Order the vertices into a sequence $x_1, \dots, x_n$ and represent x_i as $x_{i1} x_{i2} \dots x_{in}$, where

for $j < i$ $x_{ij} = 1$ if $\{x_i, x_j\} \in E(G)$, $x_{ij} = 2$ otherwise,

$x_{ii} = 0$, and

for $j > i$ $x_{ij} = 1$.

(b) Take $G = (n \times 2, \{\{(i_1,i_2),(j_1,j_2)\} \mid i_1 \neq j_1\}$ and denote by $v(i_1,i_2)$ the vector representing (i_1,i_2) in $P^{\boxtimes m}$. For $i \in n$ there has to be an $s = s(i)$ such that $|v_s(i,0) - v_s(i,0)| = 2$. Since for $j \neq i$ $v(j,k)$ is connected with both $v(i,0)$ and $v(i,1)$, we have to have $v_{s(i)}(j,k) = 1$. Thus $m \geq n$.

2.2. Proposition. Let $A_1,\dots,A_k \subset V(G)$ be discrete subsets such that whenever $\{x,y\} \notin E(G)$, there is an i with $x,y \in A_i$. Then

$$\sigma(G) \leq \sum_{i=1}^{k} \log^+ |A_i| .$$

Proof. Put $n_i = \log^+ A_i$ and consider a one-to-one mapping $v_i : A_i \to \{0,2\}^{n_i}$. For $x \notin A_i$ put, moreover, $v_i(x) = \tilde{1}(n_i)$. Now, we can embed G into $P^{\boxtimes n}$, where $n = \Sigma\, n_i$, representing x by $v_1(x)v_2(x)\dots v_k(x)$.

2.3. Proposition. $\sigma(K_n) = \sigma(D_n) = \log^+ n$.

Proof. Obviously $\sigma(K_n) = \log^+ n \leq \sigma(D_n)$. Now, consider a discrete subset D of P^k. For $v \in D$ define v' by replacing the 1-coordinates by zeros. Obviously, $|\{v' | v \in D\}| = |D|$ so that $|D| \leq 2^k$.

2.4. Lemma. Denote X_1 the set of vectors from $P_2^{\boxtimes n}$ having at least one 1-coordinate. Let $D \subset X_1$ be a discrete subset. Then $|D| \leq 2^{n-1}$.

Proof. Obviously we can assume that each element of D has exactly one 1-coordinate (replacing all the 1-coordinates but one in each $x \in D$ by zeros to get x', we obtain $D' = \{x' | x \in D\}$ which is discrete and equally large as D).

Put $M = \{0,2\}^{n-1}$ and consider the bipartite graph given by the partition (D,M) and the relation R where

$u_1 \dots u_n \; R \; v_1 \dots v_{n-1}$ iff for $i \leq n-1$ either $u_i = v_i$ or $u_i = 1$. Put $D_1 = \{u \in M | u_n = 1\}$, $D_2 = D \setminus D_1$. Obviously,

$$u \in D_i \Rightarrow \deg u = i.$$

Let uRv, wRv and $u \neq w$. Then $u,w \in D_2$ and $|u_n - w_n| = 2$ (otherwise $|u_i - w_i| \leq 1$ contradicting the discreteness). Consequently, for $v \in M$, $\deg v \leq 2$ and if $(u,v) \in R$ and $u \in D_1$ then deg v =

Thus, for p the number of edges meeting D_2 we have

$$p = 2(|D| - |D_1|) \leq 2(|M| - |D_1|)$$

and hence $|D| \leq |M| = 2^{n-1}$.

2.5. Proposition. (a) $\sigma(P_k) = \log^+ k$,

(b) $\sigma(C_{2k}) = \log^+ 2k$,

(c) $\log^+ 2k \leq \sigma(C_{2k+1}) \leq \log^+ 2k + 1$.

Proof. If P_k is embedded into $P_2^{\boxtimes n}$ so that the vertices $0, 1, \dots, k$ are represented by

$$w_0, w_1, \dots, w_k,$$

we can embed P_{2k} into $P_2^{\boxtimes n+1}$ as

(1) $w_0 0, w_1 0, \dots, w_{k-1} 0, w_k 1, w_{k-1} 2, \dots, w_1 2, \dots, w_1 2, w_0 2,$

C_{2k} as

$$w_0 1, w_1 P, \dots, w_{k-1} 0, w_k 1, w_{k-1} 2, \dots, w_2 2, w_1 2,$$

and, finally, C_{k+1} as

$$w_0 1, w_1 0, \dots, w_{k-1} 0, w_k 1, \tilde{1}(n) 2.$$

Consequently we see that

(2) $\sigma(P_k) \leq \log^+ k,\ \sigma(C_{2k}) \leq \log^+ k+1,\ \sigma(C_{2k+1}) \leq \log^+ 2k+1.$

Since all the neighbours of points from $P^{\boxtimes n} \setminus X_1$ (see 2.4) are joined with each other, all the points representing points of cycles of length > 3 and all the inner points of paths have to be in X_1. Thus, considering the sets of points corresponding to 1,3,5, etc. from P_k resp. C_k we obtain by 2.4

$$\log^+ k \leq \sigma(C_{2k}) - 1,\ \log^+ k \leq \sigma(C_{2k+1}) - 1$$

which, together with (2), yields immediately (b) and (c). For the paths we obtain so far

(3) $\log^+[k/2] + 1 \leq \sigma(P_k) \leq \log^+ k.$

The inequality $\log^+[k/2] + 1 < \log^+ k$ holds only for $k = 2^n + 1$, so that for finishing the proof of (a) it suffices to show that we cannot embed $P(2^n+1)$ into $P_2^{\boxtimes n}$.

If we could, however, the representation from (1) would yield an embedding of $P(2^{n+1}+2)$ into $P_2^{\boxtimes n}$, which already contradicts (3).

<u>Remark</u>: The formula for $\sigma(P_k)$ has been proved by another method (and in a slightly different context) in [11].

2.6. <u>Proposition</u>. Let G be a forest. Then

$$\sigma(G) \leq 4\log^+|G|.$$

<u>Proof</u>. Choose a vertex in each of the components, let

$$x_1^o, x_2^o, \dots, x_{k_o}^o$$

be the chosen points. Order the points having the distance from some of the x_j^o exactly i into a sequence

$$x_o^i, x_2^i, \dots, x_{k_i}^i.$$

Finally, denote by $\varphi_i(j)$ the unique index k such that x_j^i is joined with x_k^{i-1}.

Consider a sequence

$$w_o, w_1, \dots, w_r$$

of points of $P_2^{\boxtimes n}$ representing the path P_r with r the maximum possible distance of an $x \in G$ from some x_i^o, and a system

$$u_o, u_1, \dots, u_k \quad (k = \max_i k_i)$$

of distinct elements of $\{0,2\}^s$ where $s = \log^+ k$.

Now, we can embed G into $P^{\boxtimes r+3s}$ representing the vertices as follows

$$x_i^o \longmapsto w_o u_i \tilde{1} u_o$$

$$x_i^1 \longmapsto w_1 u_{\varphi(i)} u_i \tilde{1}$$

$$x_i^2 \longmapsto w_2 \tilde{1} u_{\varphi(i)} u_i$$

$x_i^3 \mapsto w_3 u_i \tilde{1} u_{\varphi(i)}$

$x_i^4 \mapsto w_4 u_{\varphi(i)} u_i \tilde{1}$

etc.

2.7. Proposition. $\delta(K_m \times K_n) \leq m.\log^+ n + n.\log^+ m.$

Proof follows immediately from 2.2.

2.8. Notation. Let us denote by $\lambda(n)$ the smallest even k such that

$$\binom{k}{[k/2]} \geq n.$$

We have obviously

$$\lambda(n) \leq 2\log^+ n.$$

(In fact $\lambda(n)$ does not exceed the closest even number after $\log^+ n + \log^+\log^+ n$.)

2.9. Proposition. $\delta(K_m \square K_n) \leq \lambda(m).\log^+ n.$
(Consequently, $\delta(K_m \square K_n) < 2\log^+ m.\log^+ n.$)

Proof. Take a one-one mapping $\varphi : m \to \{0,1\}^{\lambda(m)}$ such that each $\varphi(i)$ has equally many 1- and 0-coordinates. For $j = 0,1$ put $u(i,j) = \varphi(i) + \tilde{j}$. Obviously

(1) if $i = i'$ or $j = j'$, we have $|u_r(i,j) - u_r(i',j')| \leq 1$ for all r. For $j \in n$ choose distinct words $j_1 j_2 \ldots j_k$ in $\{0,1\}^k$ where $k = \log^+ n$. Now, for $(i,j) \in m \times n$ put

$$v(i,j) = u(i,j_1)u(i,j_2)\ldots u(i,j_k).$$

If $i = i'$ or $j = j'$, $v(i,j)$ is joined with $v(i',j')$ in $P^{\boxtimes \lambda(m).k}$ by (1). Let $i \neq i'$ and $j \neq j'$. Then (after possible exchange of (i,j) and (i',j') there is an r with $j_r = 0$ and $j'_r = 1$. By the definition of φ there is an s such that $\varphi_s(i) = 0$ and $\varphi_s(i') = 1$. Thus, $u_s(i,j_r) = 0$ and $u_s(i',j') = 2$.

§ 3. σ and operations with graphs

3.1. Notation: The symbols $\times$, $\square$ and $\boxtimes$ have been explained in 1.1. The symbols $G + H$ and $\sum_{i=1}^{k} G_i$ are used for usual (weak) sum of G and H (resp. of $G_1,\dots,G_k$). The strong (Zykov) sum of G and H will be denoted by $G \oplus H$.

Finally, let $V(G) = V(H)$. Then, we denote by $G \cap H$ the graph with the same set of vertices and with the set of edges $E(G) \cap E(H)$.

3.2. Obviously we have

Proposition: $\sigma(G \boxtimes H) \leq \sigma(G) + \sigma(H)$.

3.3. Proposition: $\sigma(G \cap H) \leq \sigma(G) + \sigma(H)$.

Proof: Let G (resp. H) be embedded representing the vertices x by words $u(x)$ (resp. $v(x)$). We can represent the vertices of $G \cap H$ by $u(x)v(x)$.

3.4. Proposition: $\sigma(G \oplus H) \leq \sigma(G) + \sigma(H)$.

Proof: Let G (resp. H) be embedded into $P_2^{\boxtimes m}$ (resp. $P_2^{\boxtimes n}$) representing the vertices x by $u(x)$ (resp. $v(x)$). We can embed $G \oplus H$ into $P_2^{\boxtimes m+n}$ representing the $x \in V(G)$ by $u(x)\tilde{1}(n)$ and $y \in V(H)$ by $\tilde{1}(m)v(x)$.

3.5. Proposition. $\sigma(G + H) \leq \max(\sigma(G), \sigma(H)) + 1$. More generally, $\sigma(\sum_{i=1}^{k} G_i) \leq \max \sigma(G_i) + \log^{+} k$.

Proof. Represent the vertices x of G_i in $P_2^{\boxtimes n}$, where $n = \max_i \sigma(G_i)$, by $v_i(x)$. Choose k separated vectors $u_1,\dots,u_k$ in $P_2^{\boxtimes m}$, where $m = \log^{+} k$ (see 2.3). Now, we can embed $\sum G_i$ into $P_2^{\boxtimes n+m}$ representing $x \in G_i$ by $v_i(x)u_i$.

3.6. Proposition.

$\sigma(G \times H) \leq \sigma(G) + \sigma(H) + |G| \cdot \log^{+}|H| + |H| \log^{+} |G|$.

Proof. Obviously, $G \times H = (G \boxtimes H) \cap (K(G) \times K(H))$. Thus, the inequality follows from 3.2, 3.3 and 2.7.

3.7. <u>Proposition</u>. $\sigma(G\Box H) \le \sigma(G) + \sigma(H) + 2\log^+|G|.$ $.\log^+|H|.$

<u>Proof</u>. Obviously $G\Box H = (G \boxtimes H) \cap (K(G) \Box K(H))$. Thus, the inequality follows from 3.3 and 2.9.

§ 4. σ and bipartite dimension

4.1. Recall the definition of bid G in 1.2. One sees easily that it is the minimum n such that the vertices x can be replaced by vectors $x_1 \dots x_n$ in 0,1,1,3 such that x is joined with y iff $\forall i |x_i - y_i| = 1$.

4.2. <u>Proposition</u>. For a bipartite graph G we have

$$\sigma(G) \le 3\,\mathrm{bid}\ G.$$

Moreover, if G is connected,

$$\sigma(G) \le 2\,\mathrm{bid}\ G.$$

<u>Proof</u>. Denote by F_n the connected component of P_3^n containing 00...0. Since all the components are isomorphic, we have to prove that $\sigma(P_3^n) \le 3n$ and $\sigma(F_n) \le 2n$.

For $x \in P_3^n$ consider $\tilde{x} \in P_2^{\boxtimes 3n}$ defined by

$\tilde{x}_i = \max(x_i - 1, 0)$, $\tilde{x}_{n+i} = \min(x_i, 2)$, $(i = 1, \dots, n)$

$x_{2n+i} = 2a_i(x)$ $(i = 1, \dots, n-1)$

where $a(x) \in \{0,1\}^n$ satisfies $a_n(x) = 0$ and is situated in the same component as x.

If $|x_i - y_i| = 1$ for all i then obviously $|\tilde{x}_j - \tilde{y}_j| \le 1$ for all j. Let, on the other hand, $|\tilde{x}_j - \tilde{y}_j| \le 1$ for all j. Then first, according to the last n-1 coordinates x and y are in the same component. Consequently, as one easily sees,

(*) $x_i - y_i$ are either all even or all odd.

We cannot have $|x_i - y_i| = 2$ (if the numbers were 0 and 2, $|\tilde{x}_{n+i} - \tilde{y}_{n+i}| = 2$, if they were 1 and 3, $|\tilde{x}_i - \tilde{y}_i| = 2$). Thus, $|x_i - y_i| \leq 1$ for all i, which, according to (*) yields for every $x \neq y$ that $|x_i - y_i| = 1$ for all i.

Representing just the vertices of F_n, the coordinates x_{2n+i} can be left out.

4.3. Proposition. For a bipartite graph G we have

$$\text{bid } G \leq 2\,\delta(G).$$

Proof. Consider a bipartition (V_1, V_2) of an induced subgraph G of P_3^n. For $x \in V_1$ define

$$\bar{x}_i = \begin{cases} 0 & \text{if } x_i = 0 \\ 2 & \text{if } x_i \neq 0 \end{cases} \qquad \bar{x}_{n+i} = \begin{cases} 0 & \text{if } x_i = 2 \\ 2 & \text{if } x_i \neq 2 \end{cases}$$

$(i = 1,\dots,n)$,

and for $y \in V_2$ we define

$$\bar{y}_i = \begin{cases} 1 & \text{if } y_i \neq 2 \\ 3 & \text{if } y_i = 2 \end{cases} \qquad \bar{y}_{n+i} = \begin{cases} 1 & \text{if } y_i \neq 0 \\ 3 & \text{if } y_i = 0 \end{cases}$$

$(i = 1,\dots,n)$.

Thus, if x and y are both in V_i, we have never $|\bar{x}_i - \bar{y}_i| = 1$, and for $x \in V_1$ and $y \in V_2$ always $\bar{x}_i \neq \bar{y}_i$. Let $x \in V_1$, $y \in V_2$ and $|\bar{x}_i - \bar{y}_i| \leq 1$ for all i. This excludes the possibility $|\bar{x}_i - \bar{y}_j| = 3$ and since the difference cannot be even, we are left with $|\bar{x}_j - \bar{y}_j| = 1$ for all j.

On the other hand, for all j, $|\bar{x}_j - \bar{y}_j| = 1$. Thus, either $\bar{x}_j = 0$ and $\bar{y}_j = 1$, or $\bar{x}_j = 2$ and $\bar{y}_j = 3$, or finally $\bar{x}_j = 2$ and $\bar{y}_j = 1$. Consider $j \leq n$. In the first case $x_j = 0$ and $y_j < 2$, in the second one $x_j > 0$ and $y_j = 2$; in the third case we get

$x_j > 0$ and $y_j < 2$ so far. But if $x_j = 2$ and $y_j = 0$, one has $\bar{x}_{n+j} = 0$ and $\bar{y}_{n+j} = 3$. Thus, in any case $|x_i - y_i| \leq 1$, for all i.

Obviously $\bar{x} \neq \bar{y}$ if $x \neq y$.

§ 5. Dimension of $G \boxtimes H$. Remarks

5.1. Since $\dim D_n = 2$, $\dim K_2 = 1$ and $\dim(D_n \boxtimes K_2) = \log^+ n - 1$ (see e.g. [2]) we cannot have an upper estimate of $\dim(G \boxtimes H)$ in terms of dim G and dim H only. We are going to present an upper estimate involving also the chromatic numbers and cardinalities of independent sets.

5.2. dim G (result 1.2) is the minimum n such that there exists a one-one $u: V(G) \longrightarrow \mathbb{N}^n$ ($\mathbb{N}$ is the set of natural numbers) such that $\{x,y\} \in E(G)$ iff $\forall i \; u_i(x) \neq u_i(y)$. Leaving out the requirement of one-one, we can under circumstances do with one coordinate less; such a minimum will be denoted by $d_0 G$. As in 3.3, realizing, moreover, that the vertices will be distinguished already by the first dim G coordinates, we immediately obtain

<u>Lemma</u>. $\dim(G \cap H) \leq \dim G + d_0 H$.

5.3. <u>Lemma</u>. $\dim(K_p \boxtimes D_n) \leq p.\log^+ n$.

<u>Proof</u>. This is a fact known in another formulation (see [1]). $K_p \boxtimes D_n$ is a sum of n copies of K_p. It is easily seen by induction: We will show that $\dim(K_p \boxtimes D_{2n}) = \dim(K_p \boxtimes D_n) + (K_p \boxtimes D_n)) \leq \dim(K_p \boxtimes D_n) + p$. Let us have $(x,i) \in K_p \boxtimes D_n$ represented as $u(x,i)$. Represent the elements of the first summand in $K_p \boxtimes D_n + K_p \boxtimes D_n$ as $u(x,i)\tilde{x}(P)$, those of the second

summand as u(x,i)x (x+1)...(x+p-1) (addition mod p).

5.4. Theorem. Denote by χ the chromatic number and by α the maximum cardinality of independent set. We have
$\dim(G\boxtimes H)\leq \dim G.\dim H + \dim G.\,\chi H.\log^{+}\alpha G + \dim H.\,\chi G.\log^{+}\alpha H$.

Proof. Choose colorations $\varphi:G \to K_{\chi G}$, $\varphi':H \to K_{\chi H}$. We see easily that $G\boxtimes H = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$ where $V(\mathcal{G}_1) = V(G)\times V(H)$ and

$\{(x,y),(x',y')\}\in E(\mathcal{G}_1)$ iff $\{x,x'\}\in E(G)$ or $\{y,y'\}\in E(H)$,
$\{(x,y),(x',y')\}\in E(\mathcal{G}_2)$ iff $\{x,x'\}\in E(G)$ or $x = x'$ and $\varphi'(y) = \varphi'(y')$,
$\{(x,y),(x',y')\}\in E(\mathcal{G}_3)$ iff $\{y,y'\}\in E(H)$ or $y = y'$ and $\varphi(x) = \varphi(x')$.

Obviously $\dim \mathcal{G}_1 \leq \dim G.\ \dim H$ (let $u(x) = (u_i(x))_{i\leq \dim G}$, $v(x) = (v_i(x))_{i\leq \dim H}$ be representations of G resp. H, let $f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be one-one; it suffices to put $w_{ij}(x,y) = f(u_i(x),v_j(y))$. Since the definitions of $\mathcal{G}_2$ and $\mathcal{G}_3$ are analogous, it suffices to prove (see 5.2) that $d_0\mathcal{G}_3 \leq \dim H\cdot\chi G.\cdot\alpha H$. By 5.3 there is a $u:K_{\chi G}\boxtimes D_{\alpha H} \to \mathbb{N}^k$ with $k = \chi G.\log^{+}\alpha H$ such that

(1) if $y\neq y'$, there is, for any y, y', an r such that
$$u_r(x,y) = u_r(x',y'),$$
(2) if $x\neq x'$, then always $u_r(x,y) = u_r(x',y)$.

According to a well-known fact on dimension (see e.g. [5],[4]) there is a system $\mathcal{C}_1,\ldots,\mathcal{C}_{\dim H}$ of disjoint decompositions of V(H), $\mathcal{C}_i = \{A_i,\ldots,A_{is_i}\}$ such that A_{ij} are independent and whenever $\{y,y'\}\notin E(H)$, there is an A_{ij} with $\{y,y'\}\subset A_{ij}$. Put $\mu_i(y) = j$ for $y\in A_{ij}$ and choose mappings $\psi_i:V(H)\to \alpha H$ such that all $\psi_i|A_{ij}$ are one-one. Choose a number $\varkappa$ larger than all the numbers occurring among the $u_r(x,y)$. Now, for $(x,y) \in V(\mathcal{G}_3)$, $i \leq \dim H$ and $r \leq k$ put

$$w_{ir}(x,y) = u_r(\varphi(x), \psi_i(y)) + \varkappa \cdot \mu_i(y).$$

Let $\{(x,y),(x',y')\} \in E(\mathcal{G}_3)$. If $\{y,y'\} \in E(H)$ we have always $\mu_i(y) \neq \mu_i(y')$ so that $w_{ir}(x,y) \neq w_{ir}(x',y')$ according to the choice of $\varkappa$. If $y = y'$ and $\varphi(x) = \varphi(x')$, $w_{ir}(x,y) = w_{ir}(x',y')$ by (2). Let $\{(x,y),(x',y')\} \notin E(\mathcal{G}_3)$. Then either $y = y'$ and $\varphi(x) = \varphi(x')$ and then trivially $w_{ir}(x,y) = w_{ir}(x',y')$, or $y \neq y'$ and $\{y,y'\} \in E(H)$. Then we have an i such that $y,y' \in A_{ij}$ for some j, and hence there is an r such that $w_{ir}(x,y) = w_{ir}(x',y')$ by (1).

Remark. The product dim G, dim H in the upper estimate of dim (G H) is essential: Consider the example of $G = K_m + K_1$, $H = K_n + K_1$. Here we have dim G = m, dim H = n and dim $G \ast H \geq m.n$.

5.5. For connected bipartite graphs one has dim $G \leq$ bid $G + 1$. Thus, by 4.3 we have in this case dim $G \leq 2\,\sigma(G) + 1$. For general G, however, no very satisfactory upper estimate of dim G in terms of $\sigma(G)$ can be expected. We have by [2, Prop. 3.4] dim $(P_3^{\ast n}) \geq 2^n - 1$, so that such estimate would have to be exponential in $\sigma(G)$ and hence not substantially better than the trivial

$$\dim G \leq 3^{\sigma(G)}$$

obtained from dim $G \leq |G|$ and $\log_3 |G| \leq \sigma(G)$.

5.6. A lower estimate of dim G in terms of $\sigma(G)$ only is obviously impossible according to the inequality of $\log_3 |G| \leq \sigma(G)$.

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