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ULTRAFILTER WITH $\aleph_0$ PREDECESSORS IN RUDIN-FROLÍK ORDER

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Abstract: We describe a construction of an ultrafilter p on the set $\mathbb{N}$ of integers with countable set of types of predecessors of p in the Rudin-Frolík order. The relationship between characters of ultrafilters and Rudin-Frolík order is studied and the obtained result is used in the abovementioned construction.

Key words: Ultrafilter, type of ultrafilter, Rudin-Frolík order, character of a filter, P-point.

Classification: 04A20

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§ 0. Introduction. The main result of this paper is a proof of the following theorem.

Theorem A. There exists an ultrafilter p on the set $\mathbb{N}$ such that the set of types

$$\{\tau(q);\ q \sqsubseteq p\} \tag{0.1}$$

in the Rudin-Frolík order is isomorphic to the inverse order of the set of natural numbers.

Assuming the continuum hypothesis, this theorem has been proved by A. Louveau [7] and R.C. Solomon [12]. Our proof does not need any set-theoretical assumption and works in any reasonable set theory, e.g. in the Zermelo-Fraenkel set theory with the axiom of choice. By a slight modification we obtain also

Theorem B. There exists a sequence $\{p_n; n \in \mathbb{N}\}$ of ultrafilters such that the set of types

$$\{\tau(q);\ p_{n+1} \sqsubseteq q \sqsubseteq p_n\}$$

has cardinality $2^{\aleph_0}$ for each $n \in \mathbb{N}$ and there is no ultrafilter p smaller than each p_n, $n \in \mathbb{N}$ in the Rudin-Frolík order.

As usually in such a situation, the desired ultrafilter is constructed by the transfinite induction. Using simple reductions everything we ask from the ultrafilter being constructed is to behave well in relation to a family of sets of cardinality continuum. On each step of the transfinite induction exactly one set of this family is considered. Therefore we must not construct the ultrafilter before the continuum'th step. As far as we know there was only one useful technology for keeping the transfinite construction of an ultrafilter not to finish before continuum steps: the method of independent sets developed by K. Kunen [5],[6].

In this paper we present another method for keeping the transfinite induction not to finish very early. The method is based on a simple relationship between the character (= local weight) of points of a set and the character of points of its closure (theorem 2.1).

The paper is organized as follows. The first section contains necessary facts concerning Rudin-Frolík order. The second part studies local weights of points in $\beta\mathbb{N}$. The third part contains proofs of the theorems A and B. The fourth part is devoted to some related open problems.

§ 1. <u>Preliminaries</u>. The notations used in this paper are much as in the most recent literature, e.g. W.W. Comfort and S. Negrepontis [3], but for the reader's convenience, we shall remind some notions.

In the whole paper we shall deal with filters and ultrafilters on the set $\mathbb{N}$ of natural numbers only. The Stone-Čech compactification $\beta\mathbb{N}$ is assumed to be the set of all ultrafilters on $\mathbb{N}$. For a set $A \subseteq \mathbb{N}$, the set $s(A)$ consists of all ultrafilters containing the set A. The family $\{s(A); A \subseteq \subseteq \mathbb{N}\}$ is the clopen basis for the topology on $\beta\mathbb{N}$. A set $U \subseteq \beta\mathbb{N}$ is a neighborhood of an ultrafilter $p \in \beta\mathbb{N}$ if and only if there is a set $A \in p$ such that $s(A) \subseteq U$.

In the next, by a discrete set $X \subseteq \beta\mathbb{N}$ we always understand an infinite countable discrete subset of $\beta\mathbb{N}$. Moreover, we always assume that

(1.1) $$X = \{x_n; n \in \mathbb{N}\}$$

and $A_n, n \in \mathbb{N}$ are subsets of $\mathbb{N}$ such that

(1.2a) $$A_n \cap A_m = \emptyset \text{ for } n \neq m,$$

(1.2b) $$A_n \in x_n$$

and

(1.2c) $$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}.$$

It is well known that every homeomorphism of $\beta\mathbb{N}$ onto $\beta\mathbb{N}$ is induced by a permutation of the set $\mathbb{N}$. Two ultrafilters $p, q \in \beta\mathbb{N}$ are said to be type equivalent iff there exists a homeomorphism h of $\beta\mathbb{N}$ onto $\beta\mathbb{N}$ such that $h(p) = q$. The type of p is denoted by $\tau(p)$. Thus, $\tau(p) = \tau(q)$ if and only if p, q are type equivalent. Sometimes, the type $\tau(p)$

is simply denoted by p. We shall deal with properties of ultrafilters that are invariant for the type equivalence.

Let X be a discrete set. Then there exists a unique homeomorphism h of $\beta\mathbb{N}$ into $\beta\mathbb{N}$ such that

(1.3) $$h(n) = x_n \text{ for every } n \in \mathbb{N}.$$

If $p \in \beta\mathbb{N}$, the ultrafilter $q = h(p)$ is denoted by $\Sigma(X,p)$. The type of $\Sigma(X,p)$ does not depend on the enumeration (1.1) of the set X. One can easily see that $\Sigma(X,p)$ belongs to the closure $\overline{X}$ of the set X and that for every set $A \subseteq \mathbb{N}$ the following holds true:

(1.4) $$A \in \Sigma(X,p) \equiv \{n; A \in x_n\} \in p \equiv h^{-1}(s(A) \cap X) \in p.$$

Conversely, if $r \in \overline{X}$ then there exists a unique ultrafilter $\Omega(X,r)$ such that

(1.5) $$\Sigma(X, \Omega(X,r)) = r.$$

It is easy to see that

(1.6) $$A \in \Omega(X,r) \equiv \bigcup_{n \in A} A_n \in r.$$

If $q = \Sigma(X,p)$ for some discrete set X, we shall write $p \sqsubseteq q$. The relation $\sqsubseteq$, introduced by Z. Frolík [4] and later studied by M.E. Rudin [10], is called the Rudin-Frolík order. The basic properties of the Rudin-Frolík order are presented e.g. in [10],[11],[2]. We remind the most important for our considerations.

Let X, Y be discrete sets $X = \{x_n; n \in \mathbb{N}\}$, $Y = \{y_n; n \in \mathbb{N}\}$, $p = \Sigma(X,q)$, $r = \Sigma(Y,j)$. Then the following holds true:

(1.7) $$\text{if } p = r,\ Y \subseteq \overline{X} - X, \text{ then } j \sqsubset q.$$

Now, let $q = j$. Then

(1.8) $p \sqsubseteq r$ if and only if $\{n; x_n \sqsubseteq y_n\} \in q$.

For any ultrafilter x we have

(1.9) if $x \in \overline{X \cap Y}$ then $\Omega(X,x)$, $\Omega(Y,x)$ are type equivalent.

Speaking about types of ultrafilters we always consider uniform ultrafilters only. Thus, e.g. a uniform ultrafilter p has the minimal Rudin-Frolík type iff for any uniform ultrafilter $q \sqsubseteq p$, q is type equivalent to p. The height of a (uniform) ultrafilter p in the Rudin-Frolík order is the cardinality of smaller types, i.e.

$$\overline{\overline{\{\tau(q);\ q \sqsubseteq p\}}}.$$

Thus p is minimal if and only if its height is 1.

We recall that a uniform ultrafilter p is said to be P-point (selective) iff for any system A_n, $n \in \mathbb{N}$, $A_n \notin p$, $\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$ there exists a set $A \in p$ such that for each $n \in \mathbb{N}$, $\overline{\overline{A_n \cap A}} < \aleph_0$ ($\overline{\overline{A_n \cap A}} = 1$). Every P-point is Rudin-Frolík minimal. By K. Kunen [6], there are Rudin-Frolík minimal ultrafilters which are not P-points.

A filter $\mathcal{F}$ on $\mathbb{N}$ can be represented by the non-empty closed subset

$$s(\mathcal{F}) = \bigcap \{s(A); A \in \mathcal{F}\}$$

of $\beta\mathbb{N}$. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $s(\mathcal{F}_1) \supseteq s(\mathcal{F}_2)$. For an ultrafilter j we have $s(j) = \{j\}$.

If $\mathcal{B}$ is a family of subsets of $\mathbb{N}$ with the finite intersection property then $(\mathcal{B})$ denotes the filter generated by $\mathcal{B}$, i.e.

$$A \in (\mathcal{B}) \equiv (\exists B_1, \dots, B_n \in \mathcal{B}) \; B_1 \cap \dots \cap B_n \subseteq A.$$

If $\mathcal{F}$ is a filter, A a subset of $\mathbb{N}$ such that $\mathbb{N} - A \notin \mathcal{F}$ then $(\mathcal{F} \cup \{A\})$ is simply denoted by $(\mathcal{F}, A)$. If $\mathcal{F} = (\mathcal{B})$, $\mathcal{B}$ is closed for finite intersections, then $\mathcal{B}$ is said to be a basis of the filter $\mathcal{F}$.

A sequence $\{X_n\}_{n \in \mathbb{N}}$ of discrete subsets of $\beta\mathbb{N}$ is called a <u>discrete sequence</u> iff for every $n \in \mathbb{N}$

$$X_{n+1} \subseteq \overline{X_n} - X_n. \tag{1.10}$$

One can easily see that for a discrete sequence $\{X_n\}_{n \in \mathbb{N}}$, the sets X_n, $n \in \mathbb{N}$ are mutually disjoint, i.e.

$$X_n \cap X_m = \emptyset \text{ for } n \neq m. \tag{1.11}$$

R.C. Solomon in [12], p. 211 has shown an important technical property of discrete sequences.

<u>Lemma 1</u> <u>(R.C. Solomon)</u>. Let $\{X_n\}_{n \in \mathbb{N}}$ be a discrete sequence, p_n, p being ultrafilters such that $p = \Sigma(X_n, p_n)$ for every $n \in \mathbb{N}$. If there exists an ultrafilter q such that $q \sqsubseteq p_n$ for every $n \in \mathbb{N}$ then there exists a discrete set $Y \subseteq \bigcup_{n \in \mathbb{N}} X_n$ such that $p \in \overline{Y}$ and $\Omega(Y, p) \sqsubseteq p_n$ for every $n \in \mathbb{N}$.

If B is a subset of a topological space T then the character $\chi(B)$ is the minimal cardinal α such that there exists a system of open sets $\overline{\overline{\mathcal{A}}} = \alpha$ such that an open set V contains B if and only if $A \subseteq V$ for some $A \in \mathcal{A}$. If $\mathcal{F}$ is a filter on $\mathbb{N}$ then the character $\chi(s(\mathcal{F}))$, also simply denoted $\chi(\mathcal{F})$ is the minimal cardinality of a basis of the filter $\mathcal{F}$. Remark that a filter $\mathcal{F}$ is principal if and only if $\chi(\mathcal{F}) = 1$. If $\mathcal{F}$ is non-principal then $\chi(\mathcal{F}) \geq \aleph_0$.

A filter $\mathcal{F}$ is said to be <u>adherent</u> to a discrete sequence $\{X_n\}_{n\in\mathbb{N}}$ iff

(1.12) $$s(\mathcal{F}) \cap \bigcap_{n\in\mathbb{N}} \overline{X_n} \neq \emptyset.$$

Let Y be a discrete subset of $\bigcup_{n\in\mathbb{N}} X_n$, $\{X_n\}_{n\in\mathbb{N}}$ being a discrete sequence. A filter $\mathcal{F}$ adherent to $\{X_n\}_{n\in\mathbb{N}}$ <u>presses down the set</u> Y iff there exists a set $A \in \mathcal{F}$ and a natural number $n \in \mathbb{N}$ such that

(1.13) $$Y \cap s(A) \subseteq \bigcup_{m \neq n} X_m.$$

Similarly, the filter $\mathcal{F}$ <u>pushes out the set</u> Y iff there exists a set $A \in \mathcal{F}$ such that

(1.14) $$s(A) \cap Y = \emptyset.$$

Both properties are hereditary. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ are filters, $\mathcal{F}_1$ presses down (pushes out) a set Y then the filter $\mathcal{F}_2$ also does so.

The filter $\mathcal{F}$ <u>can press down</u> (<u>can push out</u>) the set Y iff there exists a filter $\mathcal{F}'$ such that

(1.15a) $$\mathcal{F} \subseteq \mathcal{F}',$$

(1.15b) $$\chi(\mathcal{F}') \leq \chi(\mathcal{F}) \cdot \aleph_0,$$

(1.15c) $\mathcal{F}'$ is adherent to $\{X_n\}_{n\in\mathbb{N}}$

and $\mathcal{F}'$ presses down (pushes out) the set Y.

<u>Lemma 2</u>. Let $\{X_n\}_{n\in\mathbb{N}}$ be a discrete sequence, Y being a discrete subset of the union $\bigcup_{n\in\mathbb{N}} X_n$. Let j be an ultrafilter adherent to the sequence $\{X_n\}_{n\in\mathbb{N}}$.

a) If j pushes out the set Y then $j \notin \overline{Y}$.

b) If $j \in \overline{Y}$ and j presses down the set Y then there exists a natural number m such that $\Omega(Y,j)$ and $\Omega(X_m,j)$ are type equivalent.

Proof. The part a) is trivial: if j pushes out the set Y then by (1.14) we obtain $j \notin \overline{Y}$.

Now suppose $j \in \overline{Y}$. Since j presses down the set Y there exists a natural number $n \in \mathbb{N}$ and a set $A \in j$ such that

$$Y \cap s(A) \subseteq \bigcup_{m \leq n} X_m$$

Then

$$j \in \overline{Y} \cap s(A) \subseteq \overline{\bigcup_{m \leq n} (X_m \cap Y)} = \bigcup_{m \leq n} \overline{(X_m \cap Y)},$$

i.e. j belongs to $\overline{X_m \cap Y}$ for some m. The lemma follows by (1.9).

q.e.d.

Corollary 1. Let $\{X_n\}_{n \in \mathbb{N}}$ be a discrete sequence, j being an ultrafilter adherent to it. If every discrete subset Y of $\bigcup_{n \in \mathbb{N}} X_n$ is either pushed out or pressed down by the ultrafilter j then there is no ultrafilter q such that $q \sqsubseteq \Omega(X_n,j)$ for every $n \in \mathbb{N}$.

Proof. Directly from the lemma 1 and lemma 2.

q.e.d.

§ 2. Characters and Rudin-Frolík order. Every non-trivial ultrafilter on $\mathbb{N}$ has character greater than $\aleph_0$. The Martin's axiom (see [8]) implies that every non-principal ultrafilter on $\mathbb{N}$ has character $2^{\aleph_0}$. K. Kunen [5] and J. Baumgartner and R. Laver [1] have constructed models of set theory in which $2^{\aleph_0} = \aleph_2$ and there exists a selective ultrafilter with character $\aleph_1$. B. Pospíšil [9] has shown that there exists $2^{2^{\aleph_0}}$ ultrafilters of character $2^{\aleph_0}$. K. Kunen [6] has shown that there exists a Rudin-Frolík minimal ultrafilter with character $2^{\aleph_0}$.

The character of an ultrafilter behaves well in relation to the Rudin-Frolík order. The simple and important relationship between characters and Rudin-Frolík order is expressed in the following

Theorem 1. Let X be a discrete set, p, q being non-trivial ultrafilters such that $p = \Sigma(X,q)$. Then

$$\chi(p) \geq \chi(q) \cdot \inf_{A \in q} \sup_{n \in A} \chi(x_n).$$

Proof. Let $\mathcal{B}$ be a basis of the ultrafilter p, $\overline{\overline{\mathcal{B}}} = \chi(p)$.

It is clear that the set (see (1.2))

$$\{\{n;\ B \cap A_n \in x_n\};\ B \in \mathcal{B}\}$$

is a basis for the ultrafilter $q = \Omega(X,p)$. Thus $\chi(p) \geq \chi(q)$.

Now, suppose, to get a contradiction, that there exists a set $A \in q$ such that for every $n \in A$ the character $\chi(x_n)$ is greater than $\chi(p)$. Thus the system

$$\{A_n \cap B;\ B \in \mathcal{B}\} \cap x_n$$

is not a basis of the ultrafilter x_n for any $n \in A$. Therefore, there exists a set $C_n \in x_n$ such that C_n contains no $A_n \cap B \in x_n$, $B \in \mathcal{B}$ as a subset. We denote

$$C = \bigcup_{n \in A} (C_n \cap A_n).$$

By (1.4), we have $C \in p$. Since $\mathcal{B}$ is a basis of p, there exists a set $B \in \mathcal{B}$ such that $B \subseteq C$. Since A_n's are disjoint, for every $n \in A$ we have $C \cap A_n = C_n$. (1.4) implies that the set

$$\{n;\ B \cap A_n \in x_n\} \cap A$$

belongs to the ultrafilter q, especially, there exists a natural number $n \in A$ such that $B \cap A_n \in x_n$. On the other hand

$$B \cap A_n \subseteq C \cap A_n = C_n$$

- a contradiction.

q.e.d.

It seems that the theorem 1 is in some sense the best possible. We need a notion. A set $\mathcal{H} \subseteq {}^{\mathbb{N}}\mathbb{N}$ of functioms from $\mathbb{N}$ into $\mathbb{N}$ is said to be a <u>dominating family</u> iff for every function $f \in {}^{\mathbb{N}}\mathbb{N}$ there exists a function $g \in \mathcal{H}$ such that $g(n) \geq f(n)$ for each $n \in \mathbb{N}$. There are models of set theory in which $2^{\aleph_0} > \aleph_1$ and there exists a dominating family of cardinality $\aleph_1$. Also, there are models of set theory in which every dominating family has power $2^{\aleph_0}$ (e.g. the Martin axiom implies this). Moreover, it is known that for any selective ultrafilter p there exists a dominating family $\mathcal{H}$ such that $\overline{\overline{\mathcal{H}}} \leq \chi(p)$.

Now we shall give an upper bound for the character of a produced ultrafilter.

<u>Theorem 2</u>. Let $\mathcal{H}$ be a dominating family, $\overline{\overline{\mathcal{H}}} = \Delta$. Let X be a discrete set such that every element of X is type equivalent to a given P-point p. For any ultrafilter q we have

$$\chi(\Sigma(X,q)) \leq \Delta \cdot \chi(p) \cdot \chi(q).$$

<u>Proof</u>. We suppose denotations of (1.1) and (1.2). Let $\mathcal{B}$ be a basis of p, $\overline{\overline{\mathcal{B}}} = \chi(p)$ and $\mathcal{C}$ be a basis of q, $\overline{\overline{\mathcal{C}}} = \chi(q)$.

Since each $x_n \in X$ is type equivalent to the ultrafilter p, there exists a one-to-one mapping f_n of $\mathbb{N}$ onto $\mathbb{N}$ such that

$$(2.1) \qquad x_n = \{f_n^{-1}(Z);\ Z \in p\}.$$

For given $B \in \mathcal{B}$, $C \in \mathcal{C}$, $h \in \mathcal{H}$ we denote

$$D_{B,C,h} = \bigcup_{n \in C} f_n^{-1}(B - \{0,1,\dots,h(n)\}).$$

We show that $\{D_{B,C,h}; B \in \mathcal{B}, C \in \mathcal{C}, h \in \mathcal{H}\}$ is a basis of the ultrafilter $\Sigma(X,q)$. The theorem then follows.

Let $A \in \Sigma(X,q)$. Then we have

$$E = \{n; A_n \cap A \in x_n\} \in q.$$

By (2.1), for every $n \in E$ we obtain

$$f_n(A_n \cap A) \in p.$$

Since p is a P-point there exists a set $F \in p$ such that $F - f_n(A_n \cap A)$ is finite for every $n \in E$. For $n \in E$, let $g(n)$ be the least natural number such that

$$F - \{0,1,\dots,g(n)\} \subseteq f_n(A_n \cap A).$$

From the definition of the dominating family it follows that there exists a function $h \in \mathcal{H}$ such that $g \leq h$. Since $\mathcal{B}$, $\mathcal{C}$ are bases of p, q, respectively, there are sets $B \in \mathcal{B}$, $C \in \mathcal{C}$ such that $B \subseteq F$, $C \subseteq E$. One can easily check that

$$D_{B,C,h} \subseteq A.$$

q.e.d.

Thus, if $2^{\aleph_0} > \aleph_1$, $\Delta = \chi(p) = \chi(q) = \aleph_1$, then the estimate given in the theorem 1 is the best possible. Indeed, this situation occurs in the model of [1].

In the third part we need a stronger version of the theorem 1, from which the theorem 1 actually follows. We have presented the direct proof of the theorem 1 because of its simplicity.

Lemma 3. Let X, Z be discrete subsets of $\beta\mathbb{N}$, $Z \subseteq \overline{X} - X$. Let the filter $\mathcal{F}$ be such that

(2.2) $$s(\mathcal{F}) \cap \overline{Z} \neq \emptyset$$

and for every $z \in Z$ the inequality

(2.3) $$\chi(\Omega(X,z)) > \chi(\mathcal{F})$$

holds true.

Then there exists a set $D \subseteq \mathbb{N}$ such that

(2.4) $$Z \subseteq s(D)$$

and

(2.5) $$s(\mathcal{F}) \cap \overline{X - s(D)} \neq \emptyset.$$

Proof. Suppose (1.1) and (1.2). Let $Z = \{z_n; n \in \mathbb{N}\}$, $B_n \in z_n$, $B_n \cap B_m = \emptyset$ for $n \neq m$ and $\bigcup_{n \in \mathbb{N}} B_n = \mathbb{N}$.

Let $\mathcal{B}$ be a basis of $\mathcal{F}$ such that $\overline{\overline{\mathcal{B}}} = \chi(\mathcal{F})$. We denote

$$z_n^* = \{C \subseteq X;\ (\exists A \in \mathcal{B} \cap z_n)\ s(B_n \cap A) \cap X \subseteq C\}.$$

One can easily see that z_n^* is a filter on X and that

(2.6) $$z_n^* \subseteq \{s(A) \cap X;\ A \in z_n\}.$$

The last ultrafilter is type equivalent to $\Omega(X,z_n)$, thus by (2.3) its character is greater than $\overline{\overline{\mathcal{F}}}$. Evidently $\chi(z_n^*)$ is not greater than $\overline{\overline{\mathcal{F}}}$. Therefore, in (2.6) the equality does not hold true, i.e. there exists a set $E_n \in z_n$ such that $s(E_n) \cap X \notin z_n^*$. Set

(2.7) $\quad D_n = \cup \{ A_m \cap B_n ;\ x_m \in s(E_n \cap B_n)\}$.

Then $D_n \in z_n$ and

(2.8) $\quad s(B_n) \cap s(E_n) \cap X = s(B_n) \cap s(D_n) \cap X.$

Now, denote

$$D = \bigcup_{n \in \mathbb{N}} D_n.$$

Since $D_n \in z_n$, we obtain $Z \subseteq s(D)$, i.e. (2.4) holds true.

For to prove (2.5) we let $A \in \mathcal{F}$. Since $\mathcal{B}$ is a basis of $\mathcal{F}$, there exists a set $A' \in \mathcal{B}$ such that $A' \subseteq A$. Then by (2.2) there exists $m \in \mathbb{N}$ such that $A' \in z_m$. Then also $s(A' \cap B_m) \cap X \in z_m^*$. Since $s(E_m) \cap X \notin z_m^*$, we have

$$s(A' \cap B_m) \cap X - s(E_m) \neq \emptyset.$$

By (2.7), $D_n \subseteq B_n$ and therefore (B_n's are disjoint)

$$B_n - D = B_n - D_n.$$

Using (2.8) we obtain

$$\begin{aligned} s(A) \cap (X - s(D)) &\supseteq s(A') \cap (X - s(D)) \supseteq \\ &\supseteq s(A') \cap s(B_m) \cap X - s(D) = \\ &= s(A') \cap s(B_m) \cap X - s(D_m) = \\ &= s(A' \cap B_m) \cap X - s(E_m) \neq \emptyset. \end{aligned}$$

q.e.d.

§ 3. Proof of main theorems. We start with an important technical auxiliary result.

Lemma 4. Let the filter $\mathcal{F}$ be adherent to a discrete sequence $\{X_n\}_{n \in \mathbb{N}}$. Let Y be a discrete set, $Y \subseteq \bigcup_{n \in \mathbb{N}} X_n$ such that

(3.1) $\quad \chi(\Omega(X_n, y)) > \chi(\mathcal{F})$

for every $y \in Y \cap \overline{X_n} - X_n$, $n \in \mathbb{N}$.

If $\mathcal{F}$ cannot press down the set Y then $\mathcal{F}$ can push out the set Y.

Proof. Let $B_y \subseteq \mathbb{N}$ be such that $B_y \in y$, $B_y \cap B_{y'} = \emptyset$ for $y \neq y'$, $\bigcup_{y \in Y} B_y = \mathbb{N}$.

Denote

$$Y_n = \{y \in Y;\ (\exists\, m > n)\ y \in X_m\},$$
$$V_n = \bigcup_{y \in Y_n} B_y.$$

By (1.11) we have

$$Y_n \subseteq \overline{X_n} - X_n.$$

Evidently

$$Y \cap s(-V_n) \subseteq \bigcup_{m \leq n} X_m. \tag{3.2}$$

Since the filter $\mathcal{F}$ cannot press down the set Y, then either $V_n \in \mathcal{F}$ or $(\mathcal{F}, -V_n)$ is not adherent to $\{X_k\}_{k \in \mathbb{N}}$. In both cases

$$s(-V_n) \cap s(\mathcal{F}) \cap \bigcap_{k \in \mathbb{N}} \overline{X_k} = \emptyset.$$

Hence, for every $n \in \mathbb{N}$ we have

$$s(V_n) \cap s(\mathcal{F}) \cap \bigcap_{k \in \mathbb{N}} \overline{X_k} \neq \emptyset. \tag{3.3}$$

Let $\mathcal{F}' = (\mathcal{F} \cup \{V_0, V_1, \dots, V_n, \dots\})$. Evidently, neither $\mathcal{F}'$ can press down the set Y. Hence, for each $n \in \mathbb{N}$ and for each $A \in \mathcal{F}'$ we obtain

$$Y \cap s(A) \not\subseteq \bigcup_{m \leq n} X_m. \tag{3.4}$$

If $s(A) \cap Y = \emptyset$ for some $A \in \mathcal{F}'$ then we are ready. The filter $\mathcal{F}'$ pushes out the set Y. Thus, we shall suppose that $s(A) \cap Y \neq \emptyset$ for each $A \in \mathcal{F}'$.

Then from (3.4) we obtain

$$s(A) \cap \overline{Y_n} \neq \emptyset.$$

Therefore

$$s(\mathcal{F}') \cap \overline{Y_n} \neq \emptyset.$$

Now, using the lemma 3, we get a set $D_n \subseteq \mathbb{N}$ such that

(3.5a) $\qquad Y_n \subseteq s(D_n)$,

(3.5b) $\qquad s(\mathcal{F}') \cap \overline{X_n - s(D_n)} \neq \emptyset$.

For $y \in Y \cap X_n$ we set

$$E_y = \bigcap_{m<n} B_y \cap D_m.$$

It is clear that $E_y \in y$, $\bigcup_{y \in Y_n} E_y \subseteq D_n$ and $E_y \subseteq B_y$. Now we denote

$$E = \bigcup_{y \in Y} E_y$$

and

$$\mathcal{F}'' = (\mathcal{F}', -E).$$

We start with showing that $\mathcal{F}''$ is adherent to the sequence $\{X_n\}_{n \in \mathbb{N}}$. Let $A \in \mathcal{F}''$, i.e. there exists a set $B \in \mathcal{F}'$ such that $B - E \subseteq A$. Then

$$s(A) \cap X_n \supseteq s(B \cap V_n - E) \cap X_n.$$

Since

$$B \cap V_n - E = B \cap V_n - \bigcup_{y \in Y_n} E_y,$$

we obtain

$$s(A) \cap X_n \supseteq X_n \cap s(B \cap V_n) - s(D_n).$$

This set is non-empty by (3.5b).

Since $-E \in \mathcal{F}''$ and $s(-E) \cap \overline{Y} = \emptyset$, $\mathcal{F}''$ pushes out the set Y.

Evidently $\chi(\mathcal{F}'') \leq \chi(\mathcal{F}) \cdot \aleph_0$, therefore $\mathcal{F}$ can push out the set Y.

q.e.d.

We are ready to present a proof of the main result.

<u>Proof of the theorem A</u>. We start constructing a discre-

te sequence $\{X_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$

(3.6) the height of $x \in X_n$ is n

and for every $n,m \in \mathbb{N}$, $n < m$,

(3.7) for each $x \in X_m$, $\chi(\Omega(X_n, x)) = 2^{\aleph_0}$.

According to K. Kunen [6] there exists a Rudin-Frolík minimal ultrafilter x with character $2^{\aleph_0}$. Let $X_0 = \{x_{0,k};\ k \in \mathbb{N}\}$ be a discrete set such that each $x_{0,k}$ is type equivalent to x. We define by induction

(3.8) $$x_{n+1,k} = \Sigma(X_n, x_{0,k}).$$

Let $A_{0,k} \in x_{0,k}$, $A_{0,k} \cap A_{0,k'} = \emptyset$ for $k \neq k'$ and $\bigcup_{k\in\mathbb{N}} A_{0,k} = \mathbb{N}$. By induction we set

(3.9) $$A_{n+1,k} = \bigcup\{A_{n,\ell};\ \ell \in A_{0,k}\}.$$

Using (1.4) one can easily show that $\{X_n\}_{n\in\mathbb{N}}$ is a discrete sequence. By induction, for $m > n$, it is easy to prove that $\Omega(X_n, x_{m,k})$ is type equivalent to $x_{m-n,k}$ and therefore, again by induction, theorem 1 and (3.8), we obtain (3.7).

The assertion (3.6) follows by induction and (1.8) from (3.8)

Now, let $\{Y_\xi;\ \xi < 2^{\aleph_0}\}$ be an enumeration of all discrete subsets of the union $\bigcup_{n\in\mathbb{N}} X_n$. Using the lemma 4 we can define a sequence $\{\mathcal{F}_\xi;\ \xi < 2^{\aleph_0}\}$ of filters such that

a) $\mathcal{F}_0$ is the filter of all cofinite subsets of $\mathbb{N}$;

b) $\mathcal{F}_\lambda = \bigcup_{\xi<\lambda} \mathcal{F}_\xi$ for λ limit;

c) $\mathcal{F}_{\xi+1}$ either presses down or pushes out the set Y_ξ and $\mathcal{F}_\xi \subseteq \mathcal{F}_{\xi+1}$;

d) $\chi(\mathcal{F}_\xi) \leq \overline{\overline{\xi}} \cdot \aleph_0$ for each $\xi < 2^{\aleph_0}$;

e) each $\mathcal{F}_\xi$ is adherent to $\{X_n\}_{n\in\mathbb{N}}$.

Now, let p be any ultrafilter extending the filter $\bigcup_{\xi<2^{\aleph_0}} \mathcal{F}_\xi$ and adherent to $\{X_n\}_{n\in\mathbb{N}}$. We denote $q_n = \Omega(X_n, p)$. One can easily see (using 3.6) and (1.9)) that

$$\{\tau(q);\ q \sqsubset p\} = \{\tau(q_n);\ n \in \mathbb{N}\} .$$

q.e.d.

Proof of the theorem B is almost the same as that of the theorem A just you must start with an ultrafilter x with character $2^{\aleph_0}$ and $2^{\aleph_0}$ predecessors. The existence of such an ultrafilter follows by A.K. Steiner and E.F. Steiner [13], B. Pospíšil [9] and theorem 1.

q.e.d.

Let us remark that the theorem 1 has been used in the proof of the theorem A indirectly via the lemma 4 and hence, via the lemma 3 which is a strengthening of the theorem 1.

By K. Kunen [6] there are $2^{2^{\aleph_0}}$ Rudin-Frolík minimal ultrafilters with character $2^{\aleph_0}$. Taking in the proof of the theorem A different x's we obtain different ultrafilters p's with countable set of predecessors (0.1).

§ 4. Some open problems. As far as we know that was P. Simon who raised the following question.

Problem 1. If p is a non-minimal ultrafilter, does there exist an ultrafilter q such that $\tau(q)$ $\tau(p)$ and there is no type between $\tau(q)$ and $\tau(p)$? In other words, does every non-minimal ultrafilter have an immediate predecessor?

Prof. J. Jakubík asked another question.

Problem 2. Does there exist an infimum of any finite set of type?

The question whether there exists an ultrafilter with character smaller than continuum is undecidable. Moreover, neither we knowto answer the following two questions.

Problem 3. If there exists a Rudin-Frolík minimal ultrafilter with character smaller than continuum, does there exist a non-minimal ultrafilter with such character?

Problem 4. Does the existence of an ultrafilter with character smaller than continuum imply the existence of a Rudin-Frolík minimal ultrafilter with such character?

The theorem 2 gives a partial answer to the problem 3 and the positive answer of the problem 1 implies positive answer to the problem 4.

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