Patrick Oswald

On a connectedness property of the complements of zeroneighbourhoods in topological vector spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 2, 351--356

Persistent URL: http://dml.cz/dmlcz/106081

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
22,2 (1981)

ON A CONNECTEDNESS PROPERTY OF THE COMPLEMENTS OF ZERONEIGHBOURHOODS IN TOPOLOGICAL VECTOR SPACES

P. OSWALD

Abstract. V. Klee [1] proved that for every topological vector space X there exists a base of zero-neighbourhoods $\{U\}$ whose complements are connected. More exactly, it was shown that each pair of points $x,y \in X \setminus U$ can be joined by a 8-gon contained in $X \setminus U$. In this note we give the final answer to a related question of V. Klee [1] by showing that in the above result the 8-gon's can be replaced by 2-gon's for arbitrary X.

Key words: Topological linear spaces, connected sets.

Classification: 46A15, 28A20

In this note we give the complete answer to a question of V. Klee [1] concerning connectedness properties of the complements of neighbourhoods of zero (nz) in a real separated topological vector space (tvs).

Theorem. Let X be a tvs with $\dim X \geqq 2$. Then every nz $V \subset X$ contains a nz U satisfying the following property:

(A) Each pair of points x, y belonging to the complement $X \setminus U$ of U can be joined by a 2-gon in $X \setminus U$. More exactly, there exists a point z so that $[x,z] \cup [z,y] \subset X \setminus U$.

(By $[x,y]$ we denote the line segment $\{\alpha x + (1-\alpha)y, \alpha \in [0,1]\}$, and a n-gon is defined as an arc composed of n (or fewer) line segments.)

If in the formulation of the theorem the 2-gon's are replaced by 8-gon's we get a known weaker result obtained by V. Klee in 1964 (see [1], Theorem A). In [1] he also asks whether the number 8 can be reduced for general tvs. Partial results in this direction were stated by V. Klee [1] for locally convex and locally bounded tvs, and, recently, by T. Riedrich [3] for the space S(0,1) of measurable functions. The above stated theorem improves these known results and, clearly, cannot be sharpened for any tvs X.

Proof of the theorem. Our proof is quite elementary, the reader can find the used facts from the theory of tvs and about convex sets in finite-dimensional tvs, for example, in standard sources like [4] resp. [2].

If $2 \leqq \dim X < \infty$ then X is isomorphic to the Euclidean space R^n ($X \cong R^n$) for some $n \geqq 2$. In this case the assertion of the theorem is obvious.

Let $\dim X = \infty$, and fix an arbitrary 2-dimensional (linear) subspace $E \subset X$. For any given nz $V \subset X$ the set $V \cap E$ is a nz in $E \cong R^2$. Therefore, we can find a compact convex nz $C_o \subset E$ in E, and a closed nz $V_o \subset X$ such that

(1) $E \cap V_o \subset C_o \subset E \cap V$, $V_o \subset V$.

Furthermore, let $U_o \subset X$ be a starshaped nz with

(2) $U_o + U_o + U_o + U_o \subset V_o$.

Now we define the set

(3) $U + C_o \cup (\bigcup_{F \in \mathcal{F}} \overline{\mathrm{conv}(U_o \cap F)})$

where $\mathcal{F}$ denotes the set of all 3-dimensional subspaces of X containing E (conv A stands for the convex hull of a set A, and $\overline{A}$ for the closure of A). In the following we will prove that U has the properties stated in the formulation of the theorem.

Step 1. To show that U is a nz in X with $U\subset V$ it is sufficient to verify the inclusions

(4) $U_0\subset U\subset V_0\cup C_0$ $(\subset V)$.

In virtue of (1),(2) we have $z\in C_0\subset U$ for any $z\in U_0\cap E$. If $z\in U_0\setminus E$ then define the 3-dimensional subspace $F = F(z) = \mathrm{span}(z,E)\in\mathcal{F}$ containing z. We have $z\in F\cap U_0$, and (3) implies $z\in U$. Therefore, $U_0\subset U$ has been proved.

The inclusion $U\subset V_0\cup C_0$ will be obtained by demonstrating the relation $\overline{\mathrm{conv}(U_0\cap F)}\subset V_0$ for arbitrary $F\in\mathcal{F}$. It is well-known that every point $x\in\mathrm{conv}\,M$, where M is a set in R^3, can be represented by the convex combination of at most 4 points of M (Carathéodory's theorem for R^3, see [2], p. 23). Because $F\cap U_0$ is a starshaped set in $F\cong R^3$, now it follows by (2) that $\overline{\mathrm{conv}(U_0\cap F)}\subset\overline{(F\cap U_0)+(F\cap U_0)+(F\cap U_0)+(F\cap U_0)}\subset$ $\subset\overline{V}_0 = V_0$.

Thus (4) is completely proved.

Step 2. The following relations are obvious from the construction of the nz U (see (1) - (3)): If $F\in\mathcal{F}$ then

(5) $U\cap F = \overline{\mathrm{conv}(U_0\cap F)}\cup C_0$.

Furthermore, we have

(6) $U\cap E = C_0$.

Step 3. For an arbitrary but fixed point $x\in X\setminus U$ define the set

(7) $A = A(x) = \{z\in E: [x,z]\subset X\setminus U\}$ $(\subset E)$.

The aim of the considerations of the steps 3 and 4 is to show that there are two linear independent functionals g_i belonging to the topological dual space E^* of E, and real numbers

β_i <u>such that</u>

(8) $[g_i > \beta_i] = \{z \in E : g_i(z) > \beta_i\} \subset A = A(x)$, $i = 1,2$.

In this step we consider the case when $x \in (X \setminus U) \cap E$. From (6) it follows that $x \notin C_0$. Because $C_0 \subset E \cong R^2$ is compact and convex, the set

$B = \{g \in E^* : g(x) > g(y) \text{ for all } y \in C_0\}$

is non-empty and open in the topology of E^* ($\cong R^2$). Therefore we can find two linear independent $g_i \in B \subset E^*$, and putting $\beta_i = g_i(x)$, $i = 1,2$, we get (8). Indeed, if $z \in [g_i > \beta_i]$ then $g_i(\alpha z + (1-\alpha)x) = \alpha g_i(z) + (1-\alpha)g_i(x) \geq g_i(x) > g_i(y)$, $\alpha \in [0,1]$, for all $y \in C_0$ $(i = 1,2)$. Thus, $[x,z] \subset E \setminus C_0 \subset X \setminus U$.

<u>Step 4</u>. Now let $x \in (X \setminus U) \setminus E$. Consider the subspace $F = F(x) = \mathrm{span}(x,E) \in \mathcal{F}$. Let $f_0 \in F^*$ be the functional satisfying $\mathrm{Ker}\, f_0 = E$, and $f_0(x) = 1$. First we show that

(9) $C = \overline{\mathrm{conv}(U_0 \cap F)} \cap \{y \in F : 0 \leq f_0(y) \leq 1 = f_0(x)\}$

is a compact convex set in F. Obviously, $C \subset F \cong R^3$ is convex, closed, and starshaped (with respect to zero). If we assume that C is unbounded then it follows immediately from the mentioned properties that C contains a certain half-line beginning at zero. According to (9) this half-line belongs to $E = \mathrm{Ker}\, f_0$. Therefore, $C \cap E \subset C_0$ is unbounded, which contradicts the boundedness of C_0. Thus, C is bounded (and compact), too.

Now consider the set

$\widetilde{B} = \{f \in F^* : f(x) > f(y) \text{ for all } y \in C\}$.

Because of the established properties of C the set $\widetilde{B}$ is non-empty and open in the topology of $F^* \cong R^3$. This yields that one can find three linear independent functionals $f_j \in \widetilde{B} \subset F^*$, $j = 1,2,3$. Therefore, under the functionals $f_j|_E \in E^*$ there are two linear independent ones which we denote by g_i (without loss of generality, assume that $g_i = f_i|_E$), $i = 1,2$. Finally, the reals β_i will be chosen in such a way that we have $\beta_i \geqq f_i(x)$, and $\beta_i > g_i(y)$ for all $y \in C_0$ $(i = 1,2)$.

Hence, $[g_i > \beta_i] \cap C_i = \emptyset$, and analogously to the considerations in step 3 we get $[x,z] \subset F \setminus C$ for all points $z \in [g_i > \beta_i]$, $i = 1,2$. But from (9) and the relation (5) it becomes clear that for $z \in E$ we have $[x,z] \subset X \setminus U$ iff $z \notin C_0$ and $[x,z] \subset F \setminus C$. Therefore, (8) is completely proved.

Step 5. To finish the proof of the theorem we mention that the property (A) is equivalent to the relation $A(x) \cap \cap A(y) \neq \emptyset$, where x, y are arbitrary two points of $X \setminus U$. But this follows at once by (8) and the fact that the intersection of two half-planes $[g > \beta]$ and $[\tilde{g} > \tilde{\beta}]$ is empty, may be, in the case of linear dependent functionals g and $\tilde{g}$, only.

Thus, the theorem is proved in full detail.

Remark. As it was pointed out to us by T. Jerofsky, the following statement of, may be, independent interest holds true: Let X be a tvs with $\dim X = \infty$, and E be an arbitrary but fixed n-dimensional subspace of X $(n = 1,2,\ldots)$. Then there exists a base of zero-neighbourhoods in X whose intersections with every $(n+1)$-dimensional subspace of X containing E are convex sets. To show this it suffices to modify slightly the construction given above.

Acknowledgment. The author is indebted to T. Riedrich for suggesting him the problem, and, especially to T. Jerofsky for the help in finding a simple version of the proof of the theorem.

R e f e r e n c e s

[1] KLEE V.: Connectedness in topological linear spaces, Israel J. of Math. 2(1964), 127-131.

[2] LEICHTWEISS K.: Konvexe Mengen, Deutscher Verlag der Wissenschaften, Berlin 1980.

[3] RIEDRICH T.: On vector-topological properties of zero-neighbourhoods of topological vector spaces, Comment. Math. Univ. Carolinae 21(1980), 119-129.

[4] SCHAEFER H.: Topological vector spaces, Macmillan, New York/London 1966.

Technische Universität Dresden

Sektion Mathematik

8027 Dresden, Mommsenstr. 13

D D R

(Oblatum 14.8. 1980)