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Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 2, 215--221

Persistent URL: http://dml.cz/dmlcz/106069

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

22,2 (1981)

BOUNDED NONLINEAR PERTURBATIONS OF SECOND ORDER LINEAR ELLIPTIC PROBLEMS

Pavel DRÁBEK

Abstract: In this paper we prove the existence and multiplicity results for the Dirichlet problem

$$\begin{cases} \mathcal{L}u - \lambda_1 u - g(u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $g: \mathbb{R} \to \mathbb{R}$ is a bounded continuous function.

Key words: Nonlinear elliptic problems, multiple solutions, Leray-Schauder degree theory.

Classification: 35J65

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1. Assumptions. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Let $A: D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be a linear operator with closed range $R(A)$. Moreover the nullspace $N(A)$ of A is generated by a function $\varphi \in C^1(\overline{\Omega})$ such that $\varphi > 0$ in Ω, $\varphi = 0$ on $\partial\Omega$ and $\frac{\partial \varphi}{\partial \nu} < 0$ on $\partial\Omega$, where $\frac{\partial}{\partial \nu}$ is the outward normal derivative. We can suppose that $\int_\Omega \varphi = 1$. Let us suppose that

$$L^2(\Omega) = N(A) \oplus R(A),$$

and $K = A^{-1}$ is a well-defined operator from $R(A)$ onto $D(A) \cap R(A)$ (K is called the right inverse of A). Moreover, let us suppose

(1) $K(R(A) \cap L^p(\Omega)) \subset W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega)$, $p \geq 2$,

and

(2) $\|Kf\|_{W^{2,p}(\Omega)} \leq c_p \|f\|_{L^p(\Omega)}, \quad f \in R(A) \cap L^p(\Omega).$

<u>Example</u>. Let $\mathcal{L}$ be a second order symmetric uniformly elliptic operator with smooth coefficients acting on real valued functions defined in a bounded smooth domain Ω in $\mathbb{R}^N$. Let us denote by λ_1 the first eigenvalue of the eigenvalue problem $\mathcal{L}u = \lambda u$ in Ω , $u = 0$ on $\partial\Omega$. Then the operator

$$A : u \longmapsto \mathcal{L}u - \lambda_1 u$$

satisfies all the above conditions (see [2]).

Let us denote by P and Q the orthogonal projections on $L^2(\Omega)$ into N(A) and R(A), respectively. Then each $f \in L^2(\Omega)$ admits a decomposition

$$f = Pf + Qf = s\varphi + h, \quad s \in \mathbb{R}, h \in R(A).$$

Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous and bounded function and let $b > 0$ be such that $|g(z)| \leq b$, for all $z \in \mathbb{R}$. Then the Nemytskii operator

$$G(u)(x) = g(u(x))$$

is bounded and continuous in every $L^p(\Omega)$, $1 \leq p \leq +\infty$, and

$$\|G(u)\|_{L^p(\Omega)} \leq b\,(\text{meas}\,\Omega)^{1/p}, \quad u \in L^p(\Omega).$$

2. <u>Main result</u>. Let us suppose that there exist real numbers T_1 and T_2 such that the function g satisfies the following condition:

(g) for any given $R > 0$ there is $t_0 > 0$ such that

$$\int_\Omega g(u)\varphi \geq T_2, \quad \int_\Omega g(v)\varphi \leq T_1, \quad T_1 \leq T_2,$$

for all $u, v \in L^2(\Omega)$ with $u \geq t_0\varphi$, $v \leq -t_0\varphi$, $\|Qu\|_{L^2(\Omega)} \leq R$,

$\|Qu\|_{L^2(\Omega)} \leq R.$

<u>Theorem 1</u>. Let A and g satisfy all the above assumptions, $f \in L^{\infty}(\Omega)$. Then the equation

$$A(u) + G(u) = f \tag{3}$$

has a solution $u \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p \geq 2$, provided $s\|\varphi\|_{L^2(\Omega)} \in \langle T_1, T_2 \rangle$.

<u>Proof</u>. The equation (3) is equivalent to the bifurcation system

$$v + KQG(t\varphi + v) - KQf = 0, \tag{4i}$$

$$PG(t\varphi + v) - Pf = 0, \tag{4ii}$$

$u = t\varphi + v$, $v \in R(A)$, $t \in \mathbb{R}$.

Let $v \in R(A)$ be an eventual solution of (4i) for arbitrary but fixed $t \in \mathbb{R}$. Since $g(t\varphi + v) \in L^{\infty}(\Omega)$ it follows from (1) that $v \in W^{2,p}(\Omega) \cap W_0^{1,2}(\Omega)$ for all p, and from (2) we obtain

$$\|v\|_{W^{2,p}(\Omega)} \leq c_p \, [\, b(\text{meas}\,\Omega)^{1/p} + \|Qf\|_{L^{\infty}(\Omega)}] .$$

By the Sobolev imbedding theorem (see e.g. [6]) it is $v \in C^{1,\alpha}(\overline{\Omega})$ and

$$\|v\|_{C^{1,\alpha}(\overline{\Omega})} \leq \text{const.} = c.$$

Let us remark that the constants c_p, c do not depend on $t \in \mathbb{R}$.

By a standard application of the Leray-Schauder degree theory it is possible to prove that for fixed $t \in \mathbb{R}$ there is at least one $v \in R(A)$ satisfying (4i). Let us denote

$S = \{(t,v) \in \mathbb{R} \times R(A);\ v \text{ satisfies } (4i)\}.$

Then the solutions of (3) are exactly such $u = t\varphi + v$ that $(t,v) \in S$ and $\psi(t,v) = (f,\varphi)_{L^2(\Omega)}$, where $\psi(t,v) = \int_{\Omega} g(t\varphi + v)\varphi$ is a real function defined on S. According to (g) there exists such a $t_1 > 0$ that

(5) $\int_\Omega g(t_1\varphi + v)\varphi \geq T_2$, $\int_\Omega g(-t_1\varphi + w)\varphi \leq T_1$,

for all $(t_1,v) \in S$ and $(-t_1,w) \in S$.

According to Lemma 1.2 from [1] there exists a connected subset $S_1 \subset S$ such that $\langle -t_1,t_1\rangle \subset \mathrm{proj}_{\mathbb{R}} S_1$. The function $\psi: S_1 \to \mathbb{R}$ is bounded, continuous and according to (5) $\psi(t_1,v) \geq (f,\varphi)$, $\psi(-t_1,w) \leq (f,\varphi)$, for all $(t_1,v) \in S_1$, $(-t_1,w) \in S_1$. These facts imply the existence of $t \in \langle -t_1,t_1\rangle$ and $v \in R(A)$ such that $u = t\varphi + v$ is the solution of (3). This fact completes the proof of Theorem 1.

<u>Remark 1</u>. The assertion of Theorem 1 includes Theorem 1 from [3]. Let us suppose

$(g_\pm)$ $$\lim_{z\to\pm\infty} g(z) = g_\pm\ ,\ g_- \leq g_+ \ .$$

For fixed $t \in \mathbb{R}$ and $h = Qf \notin R(A)$ put

$$\underline{t} = \inf_{(t,v)\in S} \psi(t,v),\ \overline{t} = \sup_{(t,v)\in S} \psi(t,v).$$

Then $\overline{T}_1 = \inf_{t\in\mathbb{R}} \overline{t}$, $\overline{T}_2 = \sup_{t\in\mathbb{R}} \underline{t}$ and let us suppose

(6) $$\overline{T}_1 < \overline{T}_2.$$

<u>Theorem 2</u>. Let us suppose the same as in Theorem 1 and moreover $(g\ \ )$,(6). Then for $f = s\varphi + h$ the problem (3) has

(a) at least one solution if $s\|\varphi\|_{L^2(\Omega)} \in (\overline{T}_1,\overline{T}_2)$;

(b) at least two distinct solutions if $s\|\varphi\|_{L^2(\Omega)} \in (\overline{T}_1,g_-) \cup (g_+,\overline{T}_2)$ ($\overline{T}_1 = \overline{T}_1(h)$, $\overline{T}_2 = \overline{T}_2(h)$ depend on $h \in R(A)$).

<u>Proof.</u> Let $h \in R(A)$ be fixed. If $s\|\varphi\|_{L^2(\Omega)} \in (\overline{T}_1,\overline{T}_2)$ then there exist $t_1,t_2 \in \mathbb{R}$ such that for all $(t_1,v) \in S$, $(t_2,w) \in S$ it is $\psi(t_1,v) > s\|\varphi\|_{L^2(\Omega)}$ and $\psi(t_2,w) < s\|\varphi\|_{L^2(\Omega)}$. Then the part (a) is proved using Lemma (1.2) from 1 . If $s\|\varphi\|_{L^2(\Omega)} \in (\overline{T}_1,g_-)$ then there exists $t_1 \in \mathbb{R}$ such that $\psi(t_1,v) < s\|\varphi\|_{L^2(\Omega)}$

for all $(t_1,v) \in S$ according to the definition of $\overline{T}_1$. It is sufficient to prove the existence of $t_2, t_3 \in R$ such that $\psi(t_i,v) > s\,\|\varphi\|_{L^2(\Omega)}$ for all $(t_i,v) \in S$, i=2,3. Let us denote

$$\Omega_{n,t} = \{x \in \Omega\ ;\ t\varphi(x) + v(x) \leq n, \text{ for all } (t,v) \in S\},$$

$$\Omega_{-n,t} = \{x \in \Omega\ ;\ t\varphi(x) + v(x) \geq -n, \text{ for all } (t,v) \in S\}.$$

It is easy to see that

$$\lim_{t \to \pm\infty} \text{meas}\ \Omega_{\pm n,t} = 0$$

for each $n \in \mathbb{N}$.

For sufficiently small $\varepsilon > 0$ we can choose according to $(g_{\pm})$ such a $t_0 \in R$ that for $t_2 = -t_0$ it is

$$(7)\qquad \Big|\int_{\Omega\setminus\Omega_{-n,t_2}} g(t_2\varphi + v)\varphi - g_-\Big| < \frac{\varepsilon}{2}\,,\quad \Big|\int_{\Omega_{-n,t_2}} g(t_2\varphi + v)\varphi\Big| < \frac{\varepsilon}{2}\,,$$

and for $t_3 = t_0$ it is

$$(8)\qquad \Big|\int_{\Omega\setminus\Omega_{n,t_3}} g(t_3\varphi + v)\varphi - g_+\Big| < \frac{\varepsilon}{2}\,,\quad \Big|\int_{\Omega_{n,t_3}} g(t_3\varphi + v)\varphi\Big| < \frac{\varepsilon}{2}\,,$$

for all $(t_i,v) \in S$, i = 2,3.

From (7),(8) we obtain

$$(9)\qquad \Big|\int_{\Omega} g(t_2\varphi + v)\varphi - g_-\Big| < \varepsilon \text{ and } \Big|\int_{\Omega} g(t_3\varphi + v)\varphi - g_+\Big| < \varepsilon\,,$$

for all $(t_i,v) \in S$, i = 2,3. The last inequalities (9) imply that the function ψ has the desired property. If $s\,\|\varphi\|_{L^2(\Omega)} \in (g_+, T_2)$, the proof of the part (b) is analogous.

Remark 2. In the case of the first eigenvalue the previous Theorem 2 extends the results obtained in [5].

3. Examples. Let us consider the Dirichlet problem

(10) $$\begin{cases} -\Delta u - \lambda_1 u - ue^{-u^2} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega . \end{cases}$$

It follows from our Theorem 1 (see also [3, p. 631]) that (10) has a solution for each $f \perp \varphi$. It is $g_+ = g_- = 0$ and for each $h \in R(A)$ where $A(u) = -\Delta u - \lambda_1 u$, it is $\overline{T}_1(h) < 0 < \overline{T}_2(h)$. It follows from Theorem 2 that for each $h \in R(A)$ the problem (10) has at least two distinct solutions provided $f = s\varphi + h$, $s \|\varphi\|_{L^2(\Omega)} \in (\overline{T}_1(h), 0) \cup (0, \overline{T}_2(h))$.

Let us consider the Dirichlet problem

(11) $$\begin{cases} -\Delta u - \lambda_1 u - e^{-u^2} = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega . \end{cases}$$

Then $g_+ = g_- = 0$, $\overline{T}_2(h) = 0$, $\overline{T}_1(h) < 0$, for each $h \in R(A)$. Theorem 2 implies that the problem (11) has at least two distinct solutions provided $f = s\varphi + h$, $s \|\varphi\|_{L^2(\Omega)} \in (T_1(h), 0)$.

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(Oblatum 23.10. 1980)