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A NEW CONCEPT OF SEPARATION

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Abstract: Two different notions of separation of sets by convex functionals are introduced: separation by a graph (g-separation) and separation by level sets (l-separation). For both separation properties necessary and sufficient conditions, respectively, are proved. Moreover, relations between g-separation and l-separation are given.

Key words: Separation of sets, non-convex sets, non-linear functional analysis.

Classification: 46-00, 46A40

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§ 1. Introduction. Separation theorems are known to have fundamental importance for several fields in mathematics, for instance in functional analysis, convex analysis and mathematical optimization. Such theorems about separation of convex sets, where the separation is carried out by hyperplanes or affine manifolds, are used mainly for studies in convex optimization problems and certain non-convex problems, too; but the borders of those considerations are well-known (cf. [6]). Therefore, in order to get results about global properties of more general non-convex problems we need a new way of separation.

In this paper we look into one direction of such developments that is the so-called separation by "convex functionals". In this topic some first results are given in [1], where the finite dimensional case is considered, and in [2] (without proofs). The purpose of this paper is to give a first representation of these assertions in a complete form.

§ 2. Notions and first results. Throughout this paper R denotes the field of the real numbers ordered (and topologized) in the usual way, R_+ is the set of non-negative reals. In the most of the assertions given below we use a (topological) vector space E_1 with dim $E_1 \geqq 2$ assuming that this space has a representation in the following manner: $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$, where Λ is a onedimensional subspace of E_1 which is (topologically) isomorphic to R.

A subset K of E_1 is called a cone if

$$\lambda K \subseteq K \qquad \forall \lambda \in R_+$$

holds. For a subset A of a (topological) vector space E_1 co A denotes the convex hull of A; int A and r-int A denote the set of all interior points of A and the set of all interior points of A with respect to ^1A, respectively, where ^1A is the affine hull of A (equipped with the topology which is induced by E_1); the closed hull of A is denoted by cl A.

With respect to separation of sets we need the following definition.

Definition. Let A and B be non-empty subsets of a vector space E_1 with dim $E_1 \geqq 2$ and $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$.

(1) A and B are said to be (proper) separable by a graph (abbreviated: g-separable) if there is a convex functional $f: E \longrightarrow R$ such that

$$(2.1)\quad \begin{cases} A \subseteq \{(x, \lambda\bar{k}) \in E \times \Lambda \,/\, f(x) \leqq \lambda\}, \\ B \subseteq \{(x, \lambda\bar{k}) \in E \times \Lambda \,/\, f(x) \geqq \lambda\} \end{cases}$$

(and $f(x) \neq \lambda$ for at least one $(x, \lambda\bar{k}) \in A \cup B$ in the case of proper separation).

(2) A and B are said to be (proper) separable by a level set (abbreviated: <u>l-separable</u>) if there is a convex functional $\hat{f}: E_1 \longrightarrow R$ such that

$$(2.2)\quad \begin{cases} A \subseteq \operatorname{epi} \hat{f}, \\ B \subseteq \operatorname{hypo} \hat{f} \end{cases}$$

(and $(y,0) \notin \operatorname{graph} \hat{f}$ for at least one $y \in A \cup B$ in the case of proper separation), where epi f and hypo f denote the epigraph and the hypograph of f, respectively.

Clearly, (2.1) is equivalent to

$$(2.1')\quad \begin{cases} f(x) \leqq \lambda & \forall (x, \lambda\bar{k}) \in A, \\ f(x) \geqq \lambda & \forall (x, \lambda\bar{k}) \in B; \end{cases}$$

(2.2) is equivalent to

$$(2.2')\quad \begin{matrix} \hat{f}(y) \leqq 0 & \forall y \in A, \\ \hat{f}(y) \geqq 0 & \forall y \in B. \end{matrix}$$

Moreover, it is easy to see that the following assertion is true (by use of $\hat{f}(y) := f(x) - \lambda$, $y = (x, \lambda\bar{k})$, $(x, \lambda\bar{k}) \in E_1$).

<u>Lemma 1</u>. If A and B are g-separable subsets of a vector space $E_1 \xleftrightarrow{\sim} E \times \Lambda$, then A and B are l-separable.

In order to show that for certain sets their g-separability follows from l-separability we need the next lemma.

<u>Lemma 2</u>. Let A and B be non-empty subsets of a topological vector space E_1, let K be a convex cone in E_1 with $\operatorname{int} K \neq \emptyset$.

(1) If A+K and B are l-separable by $\hat{f}$, then A+K and B-K are

1-separable by the same $\hat{f}$.

(2) If A+K and B are proper 1-separable by $\hat{f}$, then A+K and B-K are proper 1-separable by the same $\hat{f}$.

<u>Proof</u>. Since (2) follows from (1), we shall prove (1). By assumption we obtain $\mathrm{int}(A+K) \neq \emptyset$, $\mathrm{int}(B-K) \neq \emptyset$ and, therefore,

$$(2.3) \qquad {}^{1}(A+K) = {}^{1}(B-K) = E_1 .$$

Using that fact and $\mathrm{int}\, K \neq \emptyset$, for any $b-k_2$ with $b \in B$, $k_2 \in K \setminus \{0\}$ there exist $y \in B$ and $a+k_1$ with $a \in A$, $k_1 \in K$ and $\lambda \in (0,1)$ such that

$$(2.4) \qquad y = \lambda(a+k_1) + (1-\lambda)(b-k_2).$$

By assumption we have for a convenient convex functional $\hat{f}$

$$(2.5) \qquad \hat{f}(a+k_1) \leqq 0 \leqq \hat{f}(b) \qquad \forall a \in A,\ \forall b \in B,\ \forall k_1 \in K.$$

Applying (2.4) and convexity of $\hat{f}$, then

$$0 \leqq \hat{f}(y) \leqq \lambda \hat{f}(a+k_1) + (1-\lambda)\hat{f}(b-k_2) \leqq (1-\lambda)\hat{f}(b-k_2)$$

follows. Hence

$$0 \leqq \hat{f}(b-k_2) \qquad \forall b \in B,\ \forall k_2 \in K.$$

Together with (2.5) this proves the assertion.

<u>Lemma 3</u>. Let A and B be non-empty subsets of a topological vector space $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$ for a convenient onedimensional subspace $\Lambda \subseteq E_1$, let K be a convex cone in E_1 with $\mathrm{int}\, K \neq \emptyset$. If A+K and B are 1-separable by a continuous convex functional $\hat{f}$ such that $\hat{f}(a_0) < 0$ for at least one $a_0 \in A$, then A and B are proper g-separable.

<u>Proof</u>. We choose $\bar{k} \in \mathrm{int}\, K$ and put $\Lambda := \{\lambda \bar{k} / \lambda \in R\}$ such that $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$. Now we give the proof in three steps.

(1) For every $\bar{x} \in E$ there is $\lambda \in R$ such that $\hat{f}(\bar{x}, \lambda\bar{k}) = 0$:

Let $\bar{x} \in E$ any fixed. By assumption for that $\bar{x}$ there are $\lambda_1 \in R$

with

$$(\bar{x}, \lambda\bar{k}) \in A+\text{int } K \qquad \forall \lambda > \lambda_1$$

and $\lambda_2 \in R$ with

$$(\bar{x}, \lambda\bar{k}) \in B-\text{int } K \qquad \forall \lambda < \lambda_2.$$

Therefore, we get by Lemma 2

$$\text{(2.6)} \quad \hat{f}(\bar{x}, \lambda\bar{k}) \leqq 0 \leqq \hat{f}(\bar{x}, \mu\bar{k}) \qquad \forall \lambda > \lambda_1, \qquad \forall \mu < \lambda_2.$$

Using Bolzano's theorem (1) is proved.

(2) For every $\bar{x} \in E$ there is exactly one $\lambda \in R$ with $\hat{f}(\bar{x}, \lambda\bar{k}) = 0$:

Assuming there are $\bar{x} \in E, \bar{\lambda}, \bar{\bar{\lambda}} \in R$ with $\bar{\lambda} < \bar{\bar{\lambda}}$ such that

$$\hat{f}(\bar{x}, \bar{\lambda}\bar{k}) = \hat{f}(\bar{x}, \bar{\bar{\lambda}}\bar{k}) = 0.$$

Since $\hat{f}$ is convex,

$$\text{(2.7)} \quad \hat{f}(\bar{x}, \lambda\bar{k}) \leqq 0 \qquad \forall \lambda \in [\bar{\lambda}, \bar{\bar{\lambda}}],$$

$$\text{(2.8)} \quad \hat{f}(\bar{x}, \lambda\bar{k}) \geqq 0 \qquad \forall \lambda \notin [\bar{\lambda}, \bar{\bar{\lambda}}]$$

follow. As in the first part we may find $\lambda_1 \geqq \bar{\bar{\lambda}}$ such that $(\bar{x}, \lambda\bar{k}) \in A+\text{int } K$ for all $\lambda > \lambda_1$. Therefore, by (2.6) and (2.8)

$$\hat{f}(\bar{x}, \lambda\bar{k}) = 0 \qquad \forall \lambda > \lambda_1$$

and, further, by convexity of $\hat{f}$,

$$\hat{f}(\bar{x}, \lambda\bar{k}) = 0 \qquad \forall \lambda \geqq \bar{\lambda}.$$

Now we choose $\lambda_0 > \lambda_1$ such that

$$(\bar{x}, \lambda_0\bar{k}) \in A + \text{int } K \subseteq \text{int } (A + K), \; a_0 \neq (\bar{x}, \lambda_0\bar{k}).$$

Since $\hat{f}(a_0) < 0$ and $\hat{f}(x, \lambda_0\bar{k}) = 0$, on the line going through a_0 and $(\bar{x}, \lambda_0\bar{k})$ there are points $\bar{a} \in \text{int } (A + K)$ such that $f(\bar{a}) > 0$, but that is a contradiction to (2.5) and (2) is shown. Therefore,

$$N_{\hat{f}}(0) := \{(x, \lambda\bar{k}) \in E \times \Lambda / \hat{f}(x, \lambda\bar{k}) = 0\}$$

defines a functional $f: E \longrightarrow R$ by

(2.9) $f(x) - \lambda = \hat{f}(x, \lambda\bar{k}) = 0,\ x \in E,\ \lambda \in R;$

and we have

$$f(x) \leqq \lambda \qquad \forall (x, \lambda\bar{k}) \in A,$$
$$f(x) \geqq \lambda \qquad \forall (x, \lambda\bar{k}) \in B,$$
$$f(x_0) < \mu_0 \text{ for } a_0 = (x_0, \mu_0\bar{k}).$$

The proof of Lemma 3 is finished, if (3) is shown.

(3) f defined by (2.9) is convex on E:

Let $x_1, x_2 \in E$, $\mu \in (0,1)$ and $\lambda_1, \lambda_2 \in R$ such that

$$\hat{f}(x_1, \lambda_1\bar{k}) = \hat{f}(x_2, \lambda_2\bar{k}) = 0,$$

that means $\lambda_1 = f(x_1)$, $\lambda_2 = f(x_2)$. Then, by convexity of $\hat{f}$,

$$\hat{f}(\mu x_1 + (1-\mu)x_2, [\mu\lambda_1 + (1-\mu)\lambda_2]\bar{k}) \leqq \mu\hat{f}(x_1, \lambda_1\bar{k}) + $$
$$+ (1-\mu)\hat{f}(x_2, \lambda_2\bar{k}) = 0.$$

Hence

$$f(\mu x_1 + (1-\mu)x_2) \leqq \mu f(x_1) + (1-\mu)f(x_2).$$

§ 3. <u>Results on g-separability</u>. In this section we give some conditions which are necessary and sufficient, respectively, for g-separability of two sets.

<u>Theorem 3.1</u>. Let A and B be non-empty subsets of a topological vector space E_1. If A and B are g-separable, then $\text{int } A \cap B = \emptyset$ holds.

<u>Proof</u>. If $\text{int } A = \emptyset$, then that condition is satisfied. Now let $\text{int } A \neq \emptyset$. If we assume that $\bar{y} = (\bar{x}, \bar{\lambda}\bar{k}) \in \text{int } A \cap B$, where $(\bar{x}, \bar{\lambda}\bar{k}) \in E \times \Lambda$ and $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$, then (2.1) implies

$$\text{int } A \subseteq \text{int } \{(x, \lambda\bar{k}) / f(x) \leqq \lambda\},$$

that means on the one hand $f(\bar{x}) < \bar{\lambda}$. On the other hand (2.1) implies $f(\bar{x}) \geqq \bar{\lambda}$; that is a contradiction and the assertion is proved.

Corollary. Let A and B be non-empty subsets of a topological vector space E_1, and let int $A \neq \emptyset$. If A and B are g-separable by a convex functional f, then A and B are proper g-separable by f and, moreover, f is continuous.

The proof is easy by a result of Holmes (cf. [3], p. 84) and by the proof of Theorem 3.1.

Theorem 3.2. Let $E_1 \overset{\sim}{\longleftrightarrow} E \times \Lambda$ be a topological vector space, where Λ is spanned by $\bar{k}$, let A and B be non-empty subsets of E_1, and let $\bar{v} = (0, \bar{\lambda}\bar{k})$, $\bar{\lambda} \in \operatorname{int} R_+$, any fixed. If A and B are g-separable, then the following relation holds

$$\text{r-int } (A + R_+ \bar{v}) \cap B = \emptyset.$$

Proof. If r-int $(A + R_+ \bar{v}) \neq \emptyset$, then there is a subspace $\tilde{E} \subseteq E$ such that with respect to $\tilde{E} \times \Lambda$ the relation

$$\text{r-int}_{E\times\Lambda} (A + R_+ \bar{v}) = \operatorname{int}_{\tilde{E}\times\Lambda} (A + R_+ \bar{v})$$

is true. Then we have for the restriction $f_{\tilde{E}}$ of the separating functional $f: E \longrightarrow R$ on $\tilde{E}$ in fact

$$A \subseteq \tilde{A} := \{ (\tilde{x}, \lambda\bar{k}) \in \tilde{E} \times \Lambda \;/ f_{\tilde{E}}(\tilde{x}) \leqq \lambda \}$$

Therefore $A + R_+ \bar{v} \subseteq \tilde{A}$ and also

$$\operatorname{int} (A + R_+ \bar{v}) \subseteq \operatorname{int} \tilde{A}. \tag{3.1}$$

If we assume $\tilde{y} = (\tilde{x}, \tilde{\lambda}\bar{k}) \in \operatorname{int} (A + R_+ \bar{v}) \cap B$, then we obtain

$$f(\tilde{x}) = f_{\tilde{E}}(\tilde{x}) < \tilde{\lambda}$$

by (3.1). Using (2.1) we get a contradiction.

Theorem 3.3. Let A and B be non-empty subsets of a topological vector space E_1, let K be a convex cone in E_1 with int $K \neq \emptyset$ such that A+K is convex. If

$$\operatorname{int} (A + K) \cap B = \emptyset, \tag{3.2}$$

then A and B are proper g-separable by a continuous functional f.

Proof. For fixed $\overline{k} \in \operatorname{int} K$ we use $\Lambda = \{\lambda \overline{k} / \lambda \in R\}$ and $E \times \Lambda \overset{\sim}{\longleftrightarrow} E_1$ as a representation of E_1. Clearly, by assumption, for every $\overline{x} \in E$ there is $\overline{\lambda} \in R$ such that

$$(\overline{x}, \overline{\lambda}\,\overline{k}) \in A + K. \tag{3.3}$$

Now we show

$$f(x) := \inf\{\lambda \in R / (x, \lambda \overline{k}) \in A+K\} > -\infty, \quad x \in E, \tag{3.4}$$

by contradiction. Assuming there is $x_0 \in E$ such that

$$(x_0, \lambda \overline{k}) \in A + K \qquad \forall \lambda < \overline{\lambda} \tag{3.5}$$

(it is possible to make this assumption by convexity of A + K).

From this

$$(x_0, \lambda \overline{k}) + K \subseteq (A + K) + K = A + K \tag{3.6}$$

follows by convexity of K. For any fixed point $\tilde{y} = (\tilde{x}, \tilde{\lambda}\overline{k}) \in E \times \Lambda$ we obtain

$$\tilde{y} = (x_0, 0) + (0, \tilde{\lambda}\overline{k}) + \frac{1}{\varepsilon}\,(\varepsilon[\tilde{x} - x_0], \overline{k}) + (0, -\frac{1}{\varepsilon}\,\overline{k}),$$

where $\varepsilon > 0$ is chosen in such a way that

$$\tilde{\lambda} - \frac{1}{\varepsilon} < \overline{\lambda} \quad \text{and} \quad (\varepsilon[\tilde{x} - x_0], \overline{k}) \in K.$$

Then, using (3.5) and (3.6), we get for $\lambda := \tilde{\lambda} - \frac{1}{\varepsilon}$

$$(\tilde{x}, \tilde{\lambda}\overline{k}) \in A + K,$$

that means $A + K = E_1$. This is a contradiction to (3.2) according to $B \neq \emptyset$. Since A + K is convex, f defined by (3.4) is convenient for g-separability of A and B.
Moreover, since this g-separation is given in fact for A + K and B, the properness and continuity follows from the Corollary.

§ 4. Results on l-separability. With respect to separation of sets by level sets of a convex functional we are able to prove two results which are closely connected with similar conditions in theorems on separation by hyperplanes.

Theorem 4.1. Let A and B be non-empty subsets of a topological vector space E_1, let A be convex and int $A \neq \emptyset$. If int $A \cap B = \emptyset$, then A and B are proper l-separable by a continuous functional $\hat{f}$, where

$$\hat{f}(a) < 0 \qquad \forall a \in \operatorname{int} A.$$

Proof. Let $a_0 \in \operatorname{int} A$ any fixed element. Then

$$0 \in \operatorname{int}(A - a_0) \text{ and } (B - a_0) \cap \operatorname{int}(A - a_0) = \emptyset.$$

Using the Minkowski-functional

$$p_{\tilde{A}}(x) := \inf\{\lambda > 0 / x \in \lambda(A - a_0)\}, \quad x \in E_1,$$

of $\tilde{A} := A - a_0$ we get

$$\hat{f}(x) := p_{\tilde{A}}(x - a_0) - 1, \ x \in E_1,$$

as a convenient functional for proper l-separability of A and B (cf. [5], p. 183 ff.).

Theorem 4.2. [1] Let A and B be non-empty subsets of a locally convex Hausdorff-vector space E_1. If

$$\operatorname{cl}(\operatorname{co} A) \neq \operatorname{cl}(\operatorname{co} B),$$

then A and B are proper l-separable by a continuous functional.

Proof. Let $y_0 \in \operatorname{cl}(\operatorname{co} A)$ any fixed element. We consider

$$\tilde{A} := \operatorname{cl}(\operatorname{co} A) - y_0, \ \tilde{B} := \operatorname{cl}(\operatorname{co} B) - y_0.$$

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1) The idea of this theorem was given by R. Hildenbrandt in [1].

Then $0 \in \tilde{A}$ and we may assume - without loss of generality - that $B-y_o$ is not contained in $\tilde{A}$. Therefore, there is $b_o \in B$ such that $b_o - y_o \notin \tilde{A}$. Then, using a well-known separation theorem (cf. [4], p. 109), there is $u \in E_1'$ such that

$$
(4.1) \qquad \begin{cases} \langle u, b_o - y_o \rangle - 1 > 0, & \\ \langle u, a - y_o \rangle - 1 \leqq 0 & \forall\, a \in \mathrm{cl}\ (\mathrm{co}\ A). \end{cases}
$$

If we define $\hat{f}$ by

$$
\hat{f}(y) := \max \{0, \langle u, y - y_o \rangle - 1\}, \quad y \in E_1,
$$

then convexity and continuity of $\hat{f}$ are clear by that definition.

Furthermore, according to (4.1), we obtain

$$
\begin{aligned}
&\hat{f}(b) \geqq 0 && \forall\, b \in B, \\
&\hat{f}(b_o) > 0 \text{ for at least one } b_o \in B, \\
&\hat{f}(a) = 0 && \forall\, a \in A
\end{aligned}
$$

that means $\hat{f}$ is convenient.

References

[1] HILDEBRANDT R.: Trennung von Mengen mittels konvexer Funktionen, Diplomarbeit, TH Ilmenau 1980.

[2] HILDEBRANDT R., NEHSE R.: Separation by convex functionals, Conf. on Math. Programming, Eisenach 1980, Abstracts (to appear).

[3] HOLMES R.B.: Geometric Functional Analysis and its Applications, Springer-Verlag, New York-Heidelberg-Berlin 1975.

[4] KANTOROWITSCH L.W., AKILOV G.P.: Functional Analysis, Nauka, Moscow 1977 (in Russian).

[5] KÖTHE G.: Topologische lineare Räume I, Springer-Verlag, Berlin-Heidelberg-New York 1966.

[6] NEHSE R.: Problems in connection with the Hahn-Banach-

Theorem, Proc. of the Summer-school "Nonlinear Analysis", Berlin 1979, Akad. Verlag (to appear).

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