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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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SOLVABILITY OF THE SUPERLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

Pavel DRÁBEK

Abstract: We prove the existence and the multiplicity of the weak solutions of the boundary value problem

$$\begin{cases} \mathcal{A}u - \lambda u + g(x,u) = f \text{ in } \Omega, \\ \qquad\qquad Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\mathcal{A}$ is the differential operator, $\lambda > \lambda_1$ (the first eigenvalue of $\mathcal{A}$) and g is superlinear.

Key words: Higher order equations, boundary value problems, Galerking approximations, Brouwer degree.

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1. Assumptions. Let us suppose that Ω is a bounded open subset of $\mathbb{R}^N$ with the boundary $\partial\Omega$. Let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and

(1) $g(x,z)$ is bounded for $z \in (-\infty, 0\rangle$ uniformly with respect to almost all $x \in \Omega$ and $g(x,z)$ is bounded below for $z \in \mathbb{R}$ uniformly with respect to almost all $x \in \Omega$;

(2) $\lim\limits_{z \to +\infty} \frac{g(x,z)}{z} = +\infty$, uniformly with respect to almost all $x \in \Omega$.

We shall seek the weak solution of the boundary value problem

$$(3) \qquad \begin{cases} \mathcal{A}u - \lambda u + g(x,u) = f \text{ in } \Omega , \\ \qquad\qquad Bu = 0 \text{ on } \partial\Omega , \end{cases}$$

where B denotes Dirichlet or Neumann boundary conditions and

$\lambda > \lambda_1$. We suppose that

$$\mathcal{A} = \sum_{|\alpha|=|\beta|=k} (-1)^{|\alpha|} D^{\alpha} (a_{\alpha\beta}(x) D^{\beta})$$

and

$$a_{\alpha\beta} = a_{\beta\alpha} \in L^{\infty}(\Omega),\ \exists \gamma > 0: \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta} \xi^{\alpha} \xi^{\beta} > \gamma |\xi|^{2m},$$
$$\forall \xi \in \mathbb{R}^N.$$

Let $V = W_0^{k,2}(\Omega)$, resp. $V = W^{k,2}(\Omega)$ if B denotes the Dirichlet, resp. the Neumann boundary conditions. Let us denote

$$a(u,v) = \int_{\Omega} \sum_{|\alpha|=|\beta|=k} a_{\alpha\beta} D^{\alpha} u D^{\beta} v.$$

Then $\mathcal{A}$, jointly with the boundary condition $Bu = 0$, defines by the position

$$(Au,v)_V = a(u,v)$$

a linear bounded self-adjoint operator of V in V with infinitely many eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$. Let us suppose that $\varphi \in V$ is the only eigenfunction corresponding to λ_1, $\varphi \in L^{\infty}(\Omega)$ and $\|\varphi\|_{L^2} = 1$.

<u>Definition</u>. Let $f \in L^1(\Omega)$. We call $u_0 \in V$ the weak solution of (3) iff

(a) $g(x,u_0(x)) \in L^1(\Omega)$,

(b) for all $v \in E$ it is $a(u_0,v) - \lambda(u_0,v)_{L^2} + (g(x,u_0),v)_{L^2} = (f,v)_{L^2}$, where $E = C_0^{\infty}(\Omega)$, resp. $E = C^{\infty}(\overline{\Omega})$ if B denotes the Dirichlet, resp. the Neumann boundary conditions.

Adding constants on both sides of the equation, we may assume in future without loss of generality that

$$g(x,z) \geq 0 \tag{4}$$

for all $z \in \mathbb{R}$ and almost all $x \in \Omega$.

The space $L^2(\Omega)$ admits the orthogonal decomposition

(5) $$L^2(\Omega) = N \oplus H,$$

where N is generated by φ . For $u = e\varphi + w$, $e \in \mathbb{R}$, $w \in H \cap V$ we set

$$\| u\|_V^2 = a(w,w) + |e|^2.$$

Let $c > 0$ be such a constant that for all $u \in V$ it is $\|u\|_{L^2} \leq \leq c \|u\|_V$.

2. Main result

Theorem 1. Let us suppose (1),(2). Then to each $h \in H$ there exist real numbers $T_1(h) \leq T_2(h)$ and a closed set $M \subset \langle T_1, T_2 \rangle$ such that $T_2 \in M$ and the problem (3) has for $f = t\varphi + h$

(i) at least two distinct weak solutions for $t > T_2$,

(ii) at least one weak solution for $t \in M$,

(iii) no weak solution for $t < T_1$.

Proof. In the proof of Theorem 1 we use the Ljapunov-Schmidt method, the Galerkin method and the Brower fixed point theorem.

For each $u \in V$ we have according to (5), $u = s\varphi + w$, $s \in \in \mathbb{R}$, $\varphi \in V$, $w \in H \cap V$. At first we shall seek, for fixed $s \in \in \mathbb{R}$, such a $w_0 \in H \cap V$ that

(a′) $g(x, s\varphi(x) + w_0(x)) \in L^1(\Omega)$,

(b′) for all $v \in E \cap H$ it is

$$a(w_0, v) - \lambda(w_0, v) + (g(x, s\varphi + w_0), v) = (f, v).$$

Lemma 1. Let

$W = \{ w \in H \cap V;\ \|w\|_V = 1, a(w,w) \leq (\lambda + 1)(w,w)\}$. Then there exists $\alpha \in (0,1)$ such that $\|w^+\|_{L^2} \geq \alpha$, for all $w \in W$ (where w^+ denotes the positive part of w).

Proof of Lemma 1. Let us suppose to the contrary that there exists $\{w_n\}_{n=1}^{\infty} \subset W$, $\lim_{n\to\infty} \|w_n^+\|_{L^2} = 0$. Then after possibly passing to the subsequences we can suppose $w_n \rightharpoonup w_0 \in H \cap V$ in V and $w_n \to w_0$ in $L^2(\Omega)$. On the other hand $\|w_n\|_{L^2} \geq$ $\geq$ const. > 0. Then $w_0 \neq 0$ and $w_0 \leq 0$ a.e. in Ω. This is a contradiction with the fact $(\varphi, w_0) = 0$.

Let us remark that from (1),(2) we obtain the existence of a constant $\beta > 0$, such that

$$g(x,z) \geq \frac{\lambda c^2}{\alpha^2} z - \beta, \tag{6}$$

for all $z \in \mathbb{R}$ and for almost all $x \in \Omega$.

Lemma 2. Let $I \subset \mathbb{R}$ be a bounded interval. Then there exists a constant $r > 0$ such that for $w \in V \cap H$, $\|w\|_V \geq r$, $s \in I$ and $g(x, s\varphi + w) \in L^1(\Omega)$ it is

$b(w,w) = a(w,w) - \lambda(w,w) + (g(x,s\varphi + w),w) - (f,w) > 0.$

Proof of Lemma 2. Let us suppose to the contrary that there exist $\{\tilde{w}_n\}_{n=1}^{\infty} \subset H \cap V$, $s_n \in I$, $g(x,s_n\varphi + \tilde{w}_n) \in L^1(\Omega)$, $\|w_n\|_V \to +\infty$ and

$$b(\tilde{w}_n,\tilde{w}_n) \leq 0, \tag{7}$$

for all $n \in \mathbb{N}$. Put $w_n = \tilde{w}_n / \|\tilde{w}_n\|_V$. From (7) we obtain

$$a(w_n,w_n) - \lambda(w_n,w_n) + \frac{1}{\|\tilde{w}_n\|_V}(g(x,s_n\varphi + \tilde{w}_n),w_n) \leq \frac{\|h\|_{L^2}}{\|\tilde{w}_n\|_V} c. \tag{8}$$

Because of (1), $\varphi \in L^\infty(\Omega)$ and the boundedness of I, there exists a constant $c_1 > 0$ such that

$$(g(x,s_n\varphi + \tilde{w}_n),w_n) \geq (g(x,s_n\varphi + \tilde{w}_n)w_n^+) - c_1. \tag{9}$$

From (8) and (9) we obtain that for $w_n \notin W$ it is

$$\frac{1}{\lambda+1}\, a(w_n,w_n) + \frac{1}{\|\tilde{w}_n\|_V}\,(g(x,s_n\varphi + \tilde{w}_n),w_n^+) - \frac{c_1}{\|\tilde{w}_n\|_V} \leq$$

$$\leq \frac{\|h\|_{L^2}}{\|\tilde{w}_n\|_V}\, c.$$

Because of $\|\tilde{w}_n\|_V \to +\infty$, the last inequality implies the existence of such $n_0 \in \mathbb{N}$ that $w_n \in W$ for $n \geq n_0$. Using (6) and (9) we can write (8) as follows

$$(8') \qquad \frac{c\,\|h\|_{L^2}}{\|\tilde{w}_n\|_V} \geq a(w_n,w_n) - \lambda(w_n,w_n) + \frac{1}{\|\tilde{w}_n\|_V} \int_\Omega \frac{\lambda c^2}{\alpha^2}(s_n\varphi +$$

$$+ \|\tilde{w}_n\|_V w_n)w_n^+ dx - \frac{1}{\|\tilde{w}_n\|_V} \int_\Omega \beta\, w_n^+\, dx - \frac{c_1}{\|\tilde{w}_n\|_V} \geq a(w_n,w_n) -$$

$$- \lambda(w_n,w_n) + \lambda c^2 - \frac{c_2}{\|\tilde{w}_n\|_V} \geq a(w_n,w_n) - \frac{c_2}{\|\tilde{w}_n\|_V},$$

where $c_2 > 0$ is some constant independent of $n \in \mathbb{N}$. But (8′) is in contradiction with $\|w_n\|_V = 1$.

Lemma 3. Let $I \subset \mathbb{R}$ be a bounded interval. Then there exists $r > 0$ such that for each $s \in I$ there exists $w_0 \in V \cap H$ satisfying (a′),(b′) and $\|w_0\|_V \leq r$.

Proof of Lemma 3. Let $s \in I$ be fixed. We shall construct the solution w_0 using the Galerkin's approximations. We choose a sequence $\{w_n\}_{n=1}^{\infty} \subset C^\infty(\Omega) \cap H$, such that for every $w \in C^\infty(\Omega) \cap H$ there is a subsequence $\{\tilde{w}_n\}_{n=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ which converges to w in the norm of V. A function $u_n \in V_n = \operatorname{span}\{w_1, w_2, \dots, w_n\}$ is called a Galerkin solution of (a′),(b′) in V_n if

$$(10) \qquad b(u_n,w) = 0 \quad \text{for all } w \in V_n.$$

Define $T_n : V_n \to V_n'$ by the relation

$$\langle T_n u, v\rangle_{V_n} = b(u,v) \text{ for all } u,v \in V_n$$

($\langle .,.\rangle_{V_n}$ denotes the duality between V_n and V_n').

According to Lemma 2 there exists $r > 0$ (depending only on $I \subset \mathbb{R}$) such that

$$(11) \qquad \langle T_n w, w\rangle_{V_n} > 0 \quad \text{for } \|w\|_V \geq r.$$

The existence of u_n follows, now, from (11) and from the Brouwer fixed point theorem (see e.g. [3]). Using the compact imbedding $V \hookrightarrow\hookrightarrow L^2(\Omega)$, we obtain the existence of such $w_o \in V \cap H$ that after possibly passing to the subsequences $u_n \rightharpoonup w_o$ in V, $u_n \longrightarrow w_o$ in $L^2(\Omega)$ and $u_n \longrightarrow w_o$ a.e. in Ω. From (10) we obtain

$$\int_\Omega |u_n g(x, s\varphi + u_n)| \leq c_3 \|u_n\|_V^2 + \|h\|_{L^2} \|u_n\|_V \leq c_4,$$

where c_3, c_4 are constants independent of n. Because of $u_n g(x, s\varphi + u_n) \longrightarrow w_o g(x, s\varphi + w_o)$ a.e. in Ω, the Fatou's lemma implies $w_o g(x, s\varphi + w_o) \in L^1(\Omega)$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that for each $\Omega' \subset \Omega$, meas $\Omega' < \delta$ it is

$$\int_{\Omega' \cap [u_n \leq k]} |g(x, s\varphi + u_n)| < \varepsilon/2 \text{ and } \frac{1}{k} \int_{\Omega' \cap [u_n > k]} |u_n g(x, s\varphi + u_n)| < \varepsilon/2.$$

Then

$$\int_{\Omega'} |g(x, s\varphi + u_n)| \leq \int_{\Omega' \cap [u_n \leq k]} |g(x, s\varphi + u_n)| + \frac{1}{k} \int_{\Omega' \cap [u_n > k]} |u_n g(x, s\varphi + u_n)| < \varepsilon .$$

Because of $g(x, s\varphi + u_n) \longrightarrow g(x, s\varphi + w_o)$ a.e. in Ω, the Vitali's theorem implies $g(x, s\varphi + w_o) \in L^1(\Omega)$ and $g(x, s\varphi + u_n) \longrightarrow g(x, s\varphi + w_o)$ in $L^1(\Omega)$. So we have

$$b(w_0,u) = 0 \quad \text{for all } u \in \bigcup_{n=1}^{+\infty} V_n.$$

For $w \in C^{\infty}(\Omega) \cap H$ we select therefore a subsequence $\{w_n\}_{n=1}^{\infty}$, $w_n \in V_n$, $w_n \longrightarrow w$ in V and get

$$b(w_0,w) = \lim_{n \to +\infty} b(w_0,w_n) = 0,$$

which proves Lemma 3.

We shall continue in the proof of Theorem 1. Let us denote

$S = \{(s,w) \in \mathbb{R} \times (H \cap V);\ w \text{ satisfies } (a'),(b')\}$,

$S_n = \{(s,w) \in \mathbb{R} \times (H \cap V_n);\ w \text{ is a Galerkin solution of } (a'),(b')\}$.

Then the weak solutions of (3) are such $u = s\varphi + w$ that $(s,w) \in S$ and

$$(12) \qquad (\lambda_1 - \lambda)s + (g(x,s\varphi + w),\varphi) = t.$$

Let us define $F: S \cup (\bigcup_{n=1}^{\infty} S_n) \longrightarrow \mathbb{R}$ by the relation

$F(s,w) = (\lambda_1 - \lambda)s + (g(x,s\varphi + w),\varphi)$ for $(s,w) \in S \cup (\bigcup_{n=1}^{+\infty} S_n)$.

Using (1),(2) it is possible to prove by the same way as in [4, p.13] that F is a continuous function on $S \cup (\bigcup_{n=1}^{\infty} S_n)$ bounded below on $S \cup (\bigcup_{n=1}^{\infty} S_n)$ and

$$(13) \qquad \lim_{s \to \pm\infty} F(s,w) = +\infty$$

uniformly with respect to w, such that $(s,w) \in S \cup (\bigcup_{n=1}^{\infty} S_n)$.

Let us denote $T_2 = \sup_{(0,w) \in S \cup (\cup S_n)} F(0,w)$. According to Lemma 3 it is $T_2 < +\infty$. Suppose $t > T_2$, there exists $s_0 \in \mathbb{R}$ such that for all $(s,w) \in S \cup (\bigcup_{n=1}^{+\infty} S_n)$ it is $\inf_{s \in (-\infty,-s_0\rangle \cup \langle s_0,+\infty)} F(s,w) > t$ (see (13)). Slightly modifying Lemma (1.2) from [1] (see also [4, p. 14]) we obtain for each $n \in \mathbb{N}$ connected subset $\overline{S}_n \subset S_n$ such that $\mathrm{proj}_{\mathbb{R}} \overline{S}_n \supset \langle -s_0, s_0 \rangle$. Then we obtain the existence of $(s_n^1,w_n) \in \overline{S}_n$, $(s_n^2,w_n) \in \overline{S}_n$, $-s_0 < s_n^1 < 0 < s_n^2 < s_0$, $\|w_n^i\|_V < r$ (where r depends only on s_o) and $F(s_n^i, w_n^i) = t$, $i=1,2$, for each $n \in \in \mathbb{N}$. After possibly passing to subsequences we can suppose that $s_n^1 \to s^1$, $s_n^2 \to s^2$ in $\mathbb{R}$ and $w_n^i \rightharpoonup w^i$ in $V \cap H$. By the same procedure as in the proof of Lemma 3 using the Fatou's lemma and the Vitali's theorem (see also [5, p. 261]) we prove that $u_1 = s^1\varphi + w^1$, $u^2 = s^2\varphi + w^2$ are the weak solutions of (3) and $u_1 \neq u_2$ (because of $t > T_2$). Let us denote $T_1 = \inf_{(s,w) \in S} F(s,w)$. If $t < T_1$ then according to the definition of the set S there is no weak solution of (3).

Let $\{t_m\}_{m=1}^{\infty} \subset \langle T_1, T_2 \rangle$, $t_m \to t_o$ in $\mathbb{R}$ and the problem (3) with the right hand side $f_m = t_m\varphi + h$ has at least one weak solution $u_m = s_m\varphi + w_m$. According to (13) and Lemma 2 we can suppose that $s_m \to s_o$ in $\mathbb{R}$ and $w_m \rightharpoonup w_o$ in $V \cap H$. Using the Fatou's lemma and the Vitali's theorem we prove that $u_o = s_o\varphi + w_o$ is the weak solution of (3) with the right hand side $f_o = t_o\varphi + h$. This proves that the set M is closed. If we take $\{t_m\}_{m=1}^{\infty} \subset \subset \langle T_2, +\infty)$, $t_m \to T_2$, we prove analogously that $T_2 \in M$ and the proof of Theorem 1 is completed.

Let us suppose that A is an elliptic differential operator of order 2m with smooth coefficients defined on Ω, $\partial\Omega$ is supposed to be also of class C^{∞}. Using Theorems (1.4.25) and (1.4.27) from [2] and the bootstrapping procedure (see [2, p. 50-51]) we obtain

<u>Theorem 2</u>. Let $f \in C^{0,\alpha}(\Omega)$, g satisfies for $N > 2m$ the growth condition

$$|g(x,s)| \leq \text{const.}(1 + |z|^{\sigma}), \text{ for } 1 < \sigma < \frac{N+2m}{N-2m},$$

for $|z|$ sufficiently large and all $x \in \Omega$. Let g be a Lipschitz continuous function of x and z. Then the weak solutions obtained in Theorem 1 are in $C^{2m,\alpha}(\Omega)$.

3. Remarks. This paper extends the results obtained in [4] and [5], where the authors consider differential operators of second order, resp. the case $\lambda = \lambda_1$.

Our Theorem 1 is an attempt to answer the question concerning the solvability of (3) if λ is an eigenvalue of (4) and $\lambda \neq \lambda_1$ (see [5, p. 255]).

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