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ON FARKAS TYPE THEOREMS

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Abstract: For linear mappings between dual pairs of real vector spaces, a Farkas type theorem is established which extends the known results to a wider class of subsets of the domain and range space. It is shown that this and related results can be derived in a unified way. As an application, a duality statement for a linear programming problem is proved.

Key words: Linear mappings and their adjoints, Farkas type theorems, duality in linear programming.

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1. Introduction. In this note, we consider the equations

(1) $$(A'^{-1}(P^o) \cap Q^o)^o = \overline{AP + Q},$$

(2) $$(P \cap A^{-1}Q)^o = \overline{P^o + A'(Q^o)},$$

(2′) $$(P \cap A^{-1}Q)^o = P^o + A'(Q^o),$$

where A denotes a weakly continuous linear mapping and P resp. Q denotes a convex subset of the domain resp. range space of A. (The exact meaning of the symbols used is explained in the following section.)

If, in particular, A maps R^n into R^m, P equals R^n_+ (the non-negative orthant in R^n), and Q is reduced to the zero element of R^m, then (since AR^n_+ is closed) equation (1) passes into

$$(A'^{-1}R^n_-)^o = AR^n_+,$$

which expresses a famous result of Farkas [9] on linear inequalities.

It is well-known that statements on the validity of the equations (1),(2) and (2′) play an important part in establishing duality results and (necessary) optimality conditions for linear and non-linear programming problems.

Theorem 1 of the present note shows that (1) is valid under weak conditions on P and Q. This known statement is included because the proof given here is very simple and because the following results will be deduced from it.

Theorem 2 is a slight generalization of a theorem, due to Schechter [16], on the solvability of sublinear inequalities. Here it is shown that this statement can be considered as a variant of Theorem 1.

The main result of this note is Theorem 3 (ii) which presents sufficient conditions for (2′) to hold, the basic hypothesis being that

$$\bar{P} \cap A^{-1}(\text{int } Q) \neq \emptyset. \tag{3}$$

This statement encompasses and extends the results obtained earlier by Kretschmer [11] and Levine-Pomerol [13]. The proof is based on a lemma showing that (3) is sufficient for the equation

$$\bar{P} \cap A^{-1}\bar{Q} = \overline{P \cap A^{-1}Q},$$

and on the known fact that (3) also implies the closedness of $P^o + A'(Q^o)$. The rest of the proof is an application of Theorem 1.

The well-known fact that if P and Q are both closed then (2) holds, has been included here (as Theorem 3 (i)) in order to demonstrate that the case of closed sets and the (more difficult) case of non-closed sets can be treated parallel to each other by reducing both to Theorem 1.

Applying Theorem 3 (ii), we finally obtain a quite general duality result for a linear programming problem (Theorem 4).

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2. Notation and terminology. Let (E,E') be a dual pair of real vector spaces. The value of the canonical bilinear functional at $(x,u)\in E\times E'$ will be denoted $\langle x,u\rangle$. Following Bourbaki [2], we shall denote by $\sigma(E,E')$ resp. $\tau(E,E')$ the weak topology resp. the Mackey topology on E with respect to (E,E').

For a nonempty subset X of E, int X denotes the $\tau(E,E')$ - interior of X and $\overline{X}$ denotes the $\sigma(E,E')$-closure of X. Recall that if X is convex, then its closure is the same for all locally convex topologies on E compatible with (E,E'); in this sense, we shall simply speak of the closure of X or say that X is closed. Furthermore,

$$X^o = \{u\in E' \mid \langle x,u\rangle \leqq 1 \text{ for each } x\in X\}$$

denotes the polar set and

$$X^* = \{u\in E' \mid \langle x,u\rangle \leqq 0 \text{ for each } x\in X\}$$

denotes the dual set (which is always a convex cone). If X itself is a cone, then $X^o = X^*$. Here X is said to be a cone if $\lambda X \subset X$ for each $\lambda \geqq 0$. A cone which is also a convex set is said to be a convex cone.

If (F,F') is another dual pair of real vector spaces and $A:E \to F$ is a weakly continuous (i.e., continuous for the weak topologies $\sigma(E,E')$ and $\sigma(F,F')$) linear mapping, then $A':F' \to E'$ denotes the adjoint of A.

Finally, R, R_+, R_-, respectively, denotes the set of all real numbers, the set of all non-negative real numbers, and the set of all non-positive real numbers.

3. Results concerning equation (1). We start with a technical lemma, the simple proof of which is left to the reader.

Lemma 1. Let (E,E') be a dual pair of real vector spaces and let X, Y be nonempty subsets of E such that at least one of them is a cone. Then

$$(X + Y)^o = X^o \cap Y^o.$$

Now it is easy to prove the following statement.

Theorem 1. Let (E,E') and (F,F') be dual pairs of real vector spaces, and let $A:E \to F$ be a weakly continuous linear mapping. Further let P be a convex subset of E and Q a convex subset of F. Suppose that at least one of the sets P, Q is a cone and that $o \in AP + Q$. Then

$$(A'^{-1}(P^o) \cap Q^o)^o = \overline{AP + Q}.$$

Proof. It is immediate that

$$A'^{-1}(P^o) = (AP)^o.$$

Using this equation, Lemma 1 and the bipolar theorem (see, e.g., [2, p. 52]), we obtain

$$(A'^{-1}(P^o) \cap Q^o)^o = ((AP)^o \cap Q^o)^o = (AP + Q)^{oo} = \overline{AP + Q},$$

and the proof is complete.

Theorem 1 has been considered, in varying generality, by several authors (see, e.g., [1],[3],[7],[8],[12],[15]). Some of them have also established conditions guaranteeing that $AP + Q$ is closed. We shall return to this question at the end of this section.

By specifying which of the sets P, Q is a cone, we can reformulate Theorem 1 in such a way that the assumption $o \in AP +$ $+ Q$ becomes dispensible.

<u>Theorem 2</u>. Let (E,E'), (F,F') and A be as in Theorem 1. Further let P be a nonempty convex subset of E and Q a nonempty convex subset of F.

(i) If P is a cone, then for each $z \in F$ the following statements are equivalent:

(a) $z \in \overline{AP + Q}$.

(b) $\langle z,v\rangle \leqq \sup_{y \in Q}\langle y,v\rangle$ for each $v \in A'^{-1}(P^*)$.

(ii) If Q is a cone, then for each $z \in F$ the following statement (b′) is equivalent to (a):

(b′) $\langle z,v\rangle \leqq \sup_{x \in P}\langle Ax,v\rangle$ for each $v \in Q^*$.

<u>Proof</u>. (i) Choose some $y_1 \in Q$. Then $Q_1 = Q - y_1$ contains the zero element. But we also have $o \in P$ since P is a cone. Hence $o \in AP + Q_1$, and we can apply Theorem 1 with Q replaced by Q_1. Now let $z \in F$ be given. Then $z \in \overline{AP + Q}$ if and only if

$z - y_1 \in \overline{AP + Q_1}$ (since always $\overline{M} + x = \overline{M + x}$), and by virtue of Theorem 1 the latter is equivalent to

(4) $[v \in A'^{-1}(P^*)$ and $\sup_{y \in Q} \langle y - y_1, v \rangle \leqq 1] \Longrightarrow \langle z - y_1, v \rangle \leqq 1.$

Since $A'^{-1}(P^*)$ is a cone on which the sublinear functional $\sup_{y \in Q} \langle y - y_1, \cdot \rangle$ is non-negative, a well-known argument shows that (4) holds if and only if

$$\langle z - y_1, v \rangle \leqq \sup_{y \in Q} \langle y - y_1, v \rangle \text{ for each } v \in A'^{-1}(P^*),$$

and this is equivalent to (b).

(ii) is proved analogously.

Using a different method, Schechter [16] established Theorem 2 for the case that E is finite dimensional, F coincides with E, A is the identity mapping of E, and P, Q are closed. He observed that his result can also be proved in an arbitrary locally convex space E. Once this is done, it is not difficult to derive the more general statement of Theorem 2. However, the proof given here is more straightforward.

Theorem 2 can be considered as a solvability criterion for an inequality system of the form

$$\langle z, v \rangle \leqq h(v) \text{ for each } v \in K,$$

where K denotes a closed convex cone in F' and h denotes the sublinear functional defined by $h(v) = \sup_{y \in Q} \langle y, v \rangle$ on the cone of all $v \in F'$ for which the supremum is finite. Some applications of this result are given in [16].

To conclude this section, we return to the question for conditions ensuring that AP + Q is closed. We shall not discuss this problem in detail; for later use, we quote the folowing criterion only.

Lemma 2. Let (E,E'), (F,F') and A be as in Theorem 1. Further let P be a nonempty closed convex subset of E and Q a nonempty closed convex subset of F. Suppose that

$$(5) \qquad (\text{int } P^{o}) \cap A'(Q^{o}) \neq \emptyset.$$

Then $AP + Q$ is closed.

In case that P, Q are cones, Lemma 2 is due to Kretschmer [11]; an alternative proof was given by Nakamura and Yamasaki [14]. In the general case, Fan [7] established the condition

$$(\text{int } P^{o}) \cap A'(\text{int } Q^{o}) \neq \emptyset$$

instead of (5). In the above form, Lemma 2 was stated without proof by Levine and Pomerol [12]. The proof of this (and a more general) statement can be carried out by modifying the proof of a closedness criterion due to Dieudonné [4]. An alternative proof can be given by first showing that AP is closed and then applying a closedness criterion of Fan [7].

4. Results concerning equations (2) and (2'). The following lemma will be crucial for our approach.

Lemma 3. Let (E,E') and (F,F') be dual pairs of real vector spaces, and let $A:E \rightarrow F$ be a weakly continuous linear mapping. Further let P be a convex subset of E and Q a convex subset of F. Suppose that $\overline{P} \cap A^{-1}(\text{int } Q) \neq \emptyset$. Then

$$\overline{P} \cap A^{-1}\overline{Q} = \overline{P \cap A^{-1}Q}.$$

Proof. We have only to show that

$$(6) \qquad \overline{P} \cap A^{-1}\overline{Q} \subset \overline{P \cap A^{-1}Q}$$

since the reverse inclusion is obvious. Let $x_o \in \overline{P} \cap A^{-1}(\text{int } Q)$

and take any $x \in \overline{P} \cap A^{-1}\overline{Q}$. Further let V be a balanced open neighbourhood of zero for the Mackey topology $\tau(E,E')$. Then there exists $\lambda \in R$ such that $0 < \lambda \leqq 1$ and $\lambda(x_0 - x) \in V$. Consequently the point $y = \lambda x_0 + (1 - \lambda)x$ belongs to $x + V$. Since $Ax_0 \in \text{int } Q$ and $Ax \in \overline{Q}$, we have

$$Ay = \lambda Ax_0 + (1 - \lambda)Ax \in \text{int } Q$$

and so $y \in A^{-1}(\text{int } Q)$. Since A, being weakly continuous, is also continuous for the Mackey topologies (see, e.g., [2, p. 104]), $A^{-1}(\text{int } Q)$ is $\tau(E,E')$-open. Hence, with respect to $\tau(E,E')$, the set $W = (x + V) \cap A^{-1}(\text{int } Q)$ is open and so a neighbourhood of y. Since x_0, $x \in \overline{P}$ and $\overline{P}$ is convex, y also belongs to $\overline{P}$. Hence there exists z with

$$z \in P \cap W \subset P \cap (A^{-1}Q) \cap (x + V).$$

This shows that $x \in \overline{P \cap A^{-1}Q}$. Thus (5) is verified and the proof of Lemma 3 is complete.

Notice that Lemma 3, appropriately reformulated, holds in an arbitrary (i.e., not necessarily locally convex) topological vector space.

Now we can prove the main result of this note.

<u>Theorem 3</u>. Let (E,E'), (F,F') and A be as in Lemma 3. Further let P be a convex subset of E and Q a convex subset of F. Suppose that at least one of the sets P, Q is a cone and that both of them contain the respective zero element.

(i) If P, Q are both closed, then

$$(P \cap A^{-1}Q)^{\circ} = \overline{P^{\circ} + A'(Q^{\circ})}.$$

(ii) If $\overline{P} \cap A^{-1}(\text{int } Q) \neq \emptyset$, then

$$(P \cap A^{-1}Q)^{\circ} = P^{\circ} + A'(Q^{\circ}).$$

Proof. We shall apply Theorem 1 with A', Q^o, P^o in place of A, P, Q respectively. Notice that A' is also weakly continuous (see, e.g., [2, p. 101]) and that, by virtue of the bipolar theorem, $Q^{oo} = \bar{Q}$, $P^{oo} = \bar{P}$. Therefore we obtain

$$(\bar{P} \cap A^{-1}\bar{Q})^o = \overline{P^o + A'(Q^o)}, \tag{7}$$

from which (i) follows immediately.

Suppose now that the hypothesis of (ii) is satisfied. Using the fact that a set and its closure have the same polar and applying Lemma 3, we then get

$$(P \cap A^{-1}Q)^o = \left(\overline{P \cap A^{-1}Q}\right)^o = (\bar{P} \cap A^{-1}\bar{Q})^o.$$

In view of (7), it only remains to show that $P^o + A'(Q^o)$ is closed, but this follows from Lemma 2 applied to A', Q^o, P^o in place of A, P, Q respectively. (Notice that Q^o and P^o, as polar sets, are closed.) This completes the proof of Theorem 3.

Theorem 3 (i) is due to Hurwicz [10] in case that P equals E and Q is a cone, an alternative proof for this case was given by Craven and Koliha [3]. For the general case, Levine and Pomerol [12] stated without proof the equation

$$(P \cap A^{-1}Q)^o = \overline{P^o + \overline{A'(Q^o)}}$$

which is easily seen to be identical with (2).

Theorem 3 (ii) was established by Kretschmer [11] for the case that, in addition, P and Q are cones. Levine and Pomerol [13] proved Theorem 3 (ii) under the assumption that P is a convex cone and Q is an open neighbourhood of zero for $\tau(F,F')$. Notice that, in this case, Q^o is $\sigma(F',F)$-compact according to the Alaoglu-Bourbaki theorem. Hence $A'(Q^o)$ is

$\sigma(E',E)$-compact and this immediately implies the closedness of $P^o + A'(Q^o)$. Thus we need not refer to Lemma 2, and the proof given above becomes especially simple. It is obvious that Theorem 3 (ii) encompasses the respective results of [11] and [13] and also applies to cases not covered there, for instance, to the case that P and Q are both non-closed cones or that P is not a cone.

As an immediate consequence of Theorem 3 (ii), we obtain the following result of Dubovitskii and Milyutin [5].

Corollary. Let (F,F') be a dual pair of real vector spaces and let $Q_0,Q_1,\ldots,Q_n$ be convex cones in F such that $Q_0 \cap \cap \operatorname{int}(\bigcap_{i=1}^{n} Q_i) \neq \emptyset$. Then

$$(\bigcap_{j=0}^{n} Q_j)^* = \sum_{j=0}^{n} Q_j^*.$$

Proof. For n = 1 the assertion follows directly from Theorem 3 (ii), and for an arbitrary integer $n > 1$ it can then be easily verified by induction.

5. Duality in linear programming. With the aid of Theorem 3 (ii), we are able to verify the following result.

Lemma 4. Let (E,E') and (F,F') be dual pairs of real vector spaces, and let $A:E \to F$ be a weakly continuous linear mapping. Further let P be a convex subset of E and Q a convex subset of F. Suppose that at least one of the sets P, Q is a cone and that both of them contain the respective zero element. Finally let $y_0 \in F$ and suppose that there exist $x_0 \in P$, $\rho_0 > 0$ satisfying $Ax_0 - \rho_0 y_0 \in \operatorname{int} Q$.

(i) For each $(u_0,\alpha) \in E' \times R$, the following statements are equivalent:

(a) $x \in P$, $\rho \in R_+$ and $Ax - \rho y_o \in Q$ imply $\langle x, u_o \rangle \leqq \alpha\rho + 1$.

(b) There exists $v \in Q^o$ satisfying $A'v - u_o \in -P^o$ and $\langle y_o, v \rangle \leqq \alpha$.

(ii) If P, Q are both cones, then for each $(u_o, \alpha) \in E' \times R$, the following statement is equivalent to (a):

(a′) $x \in P$ and $Ax - y_o \in Q$ imply $\langle x, u_o \rangle \leqq \alpha$.

Proof. (i) We shall apply Theorem 3 (ii) with E,E′,P,A, respectively, replaced by $E \times R$, $E' \times R$, $P \times R_+$, and B, where B is defined by $B(x, \rho) = Ax - \rho y_o$ for each $(x, \rho) \in E \times R$. Notice that, by assumption, $(x_o, \rho_o) \in (P \times R_+) \cap B^{-1}(\text{int } Q)$. It is evident that (a) is equivalent to

$$(u_o, -\alpha) \in ((P \times R_+) \cap B^{-1}Q)^o$$

which, by Theorem 3 (ii), holds if and only if

(8) $$(u_o, -\alpha) \in (P^o \times R_-) + B'(Q^o).$$

Since $B'v = (A'v, -\langle y_o, v \rangle)$ for each $v \in F'$, (8) is easily seen to be equivalent to (b).

(ii) Suppose that (a) is satisfied and let $x \in P$, $Ax - y_o \in Q$. Then for each $\rho > 0$, we have $\rho x \in P$ and $A(\rho x) - \rho y_o \in Q$. Hence (a) implies that for each $\rho > 0$, $\langle \rho x, u_o \rangle \leqq \alpha\rho + 1$ or $\langle x, u_o \rangle \leqq \alpha + \frac{1}{\rho}$. Thus we obtain $\langle x, u_o \rangle \leqq \alpha$, and so (a′) holds. Suppose now that the latter is true and let $x \in P$, $\rho \in R_+$, $Ax - \rho y_o \in Q$. If $\rho > 0$, then it follows immediately that $\langle x, u_o \rangle \leqq \alpha\rho < \alpha\rho + 1$. Now let $\rho = 0$. Then for each $\sigma > 0$, we have $\sigma x \in P$, $A(\sigma x) \in Q$. Furthermore, for $x_1 = \frac{1}{\rho_o} x_o$, we have $x_1 \in P$ and $Ax_1 - y_o \in Q$. For each $\sigma > 0$, we thus obtain $\sigma x + x_1 \in P$, $A(\sigma x + x_1) - y_o \in Q$ and so by virtue of (a′), $\langle \sigma x + x_1, u_o \rangle \leqq \alpha$, which implies $\langle x, u_o \rangle \leqq 0 < \alpha \cdot 0 + 1$. Hence (a) is true.

This completes the proof of the lemma.

In a known fashion (cf. Fan [8]), we can now derive a duality statement for a linear programming problem.

Theorem 4. Let (E,E'), (F,F'), A, P, Q, y_o, x_o, ρ_o be as in Lemma 4. Further let $u_o \in E'$ and set

$$G = \{ v \in F' \mid v \in Q^o,\ A'v - u_o \in -P^o \}.$$

If G is non-empty, then $\min \{\langle y_o,v\rangle \mid v \in G\}$ exists and one has

$$(9) \quad \min \{\langle y_o,v\rangle \mid v \in G\} = \sup \left\{ \frac{\langle x,u_o\rangle - 1}{\rho} \,\middle|\, x \in P,\ \rho > 0,\ Ax - \rho y_o \in Q \right\}.$$

If, in addition, both P and Q are cones, then

$$(10) \quad \min \{\langle y_o,v\rangle \mid v \in G\} = \sup \{\langle x,u_o\rangle \mid x \in P,\ Ax - y_o \in Q\}.$$

Proof. Let M denote the set of all $\alpha \in R$ satisfying (b) of Lemma 4. It is obvious that, whenever $\min M$ exists, $\min \{\langle y_o,v\rangle \mid v \in G\}$ also exists and both values are equal. We shall show that $\min M$ does exist and equals the right-hand side of (9). By Lemma 4 (i), M coincides with the set of all $\alpha \in R$ which satisfy (a) of Lemma 4. It follows that

$$(11) \quad M = \left\{ \alpha \in R \mid x \in P,\ \rho > 0,\ Ax - \rho y_o \in Q \implies \alpha \geqq \frac{\langle x,u_o\rangle - 1}{\rho} \right\}.$$

To verify (11), it suffices to consider the case $\rho = 0$ in (a) of Lemma 4. Thus let $x \in P$ and $Ax \in Q$. Since G is non-empty, we have $A'v - u_o \in -P^o$ for some $v \in Q^o$ and so, as P or Q is a cone,

$$\langle x,u_o\rangle = \langle Ax,v\rangle - \langle x,A'v - u_o\rangle \leqq 1.$$

This proves (11) which, in turn, implies that min M exists and (9) holds.

Finally, if P and Q are both cones, then it follows from what has just been proved and Lemma 4 (ii) that (10) is satisfied. This completes the proof of the theorem.

Related duality results have earlier been obtained by Duffin [6], Kretschmer [11], Fan [7],[8], Levine-Pomerol [12],[13] and others. Notice, however, that Theorem 4 constitutes a duality statement which also applies to non-closed convex cones P, Q.

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