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A NOTE ON p-ADIC COMPLETION OF TORSION FREE ABELIAN GROUPS

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Abstract: In this note some structural properties of p-adic completion of torsion free abelian groups are studied. Particularly, there is described a connection with the quasi-splitting of mixed groups.

Key words: p-adic completion of a group, quasi-isomorphism, quasi-splitting of mixed groups.

Classification: 20K20, 20K21

If G is an abelian group and p a prime then $\hat{G}_p$ denotes the p-adic completion of G. In this note we shall describe some properties of the group $\hat{A}_p$ of a torsion free group A. For example it is shown that if A is such a group then for each torsion group T a quasi-isomorphism $G \doteq \hat{A}_p \oplus T$ implies the splitting of the group G precisely if the group A/pA is finite. Thus the p-adic completion is a group-theoretical operation giving a possibility to construct splitting groups with non-splitting quasi-isomorphic images.

All groups here are supposed to be abelian and p denotes always a prime. If G is a group the symbol $G_{(p)}$ is used to denote the p-primary component of G. For other terminology and notation we refer to [2]. If $\emptyset \neq I$ is any set and G a group then G^I is the group of the vectors $x = \{g_i\}_{i \in I}$ with $g_i \in G$

for each $i \in I$; by $G^{[I]}$ ($G^{(I)}$ resp.) we shall denote its subgroup of all elements $x = \{g_i\}$ such that for each positive integer $n \in N$ the relation $g_i \in nG$ holds for almost all $i \in I$ (the equality $g_i = 0$ holds for almost all $i \in I$ resp.). Evidently we have

$$G^{(I)} \subseteq G^{[I]} \subseteq G^I.$$

For a cardinal n the groups G^n, $G^{(n)}$ and $G^{[n]}$ are constructed in a similar way. Finally recall that the p-adic completion $\hat{G}_p$ of a group G is defined by the equation $\hat{G}_p = \varprojlim_n G/p^nG$ (see [2]). The p-divisibility of G implies $\hat{G}_p = 0$.

If J_p denotes the additive group of the ring of p-adic integers Q_p^* then we are ready to prove the following lemma.

<u>Lemma 1</u>. For a free group F of the form $F = Z^{(I)}$ $(I \neq \emptyset)$ we have $\hat{F}_p \cong J_p^{[I]}$.

Proof. Any element $y \in \hat{F}_p$ may be expressed as a sequence

$y = (y_1 + pF, y_2 + p^2F, \ldots, y_k + p^kF, \ldots)$

where $y_k \in F$ and $y_k - y_l \in p^kF$ whenever $k \leq l$. Each $y_k \in F = Z^{(I)}$ is a vector $y_k = \{a_i^{(k)}\}_{i \in I}$ with $a_i^{(k)} \in Z$. Without loss of generality we may suppose

$$(1) \qquad 0 \leq a_i^{(k)} < p^k \qquad (i \in I; k \in N);$$

but then the numbers $a_i^{(k)}$ are determined uniquely by $y \in \hat{F}_p$. The relations $y_{k+1} - y_k \in p^kF$ imply

$$(2) \qquad a_i^{(k)} \equiv a_i^{(k+1)} \pmod{p^k} \qquad (i \in I; k \in N).$$

Thus if we set $\alpha_i = (a_i^{(k)})_{k=1}$ then α_i are p-adic integers. Therefore, each element $y \in \hat{F}_p$ defines an element $\varphi(y) = \{\alpha_i\} \in J_p^I$ and we get a mapping $\varphi : \hat{F}_p \to J_p^I$. Evidently, φ is an injective group homomorphism. We shall prove that $\operatorname{Im} \varphi = J_p^{[I]}$.

To see this, take $y \in \hat{F}_p$ and $\rho(y) = \{\alpha_i\}_{i\in I}$ with $\alpha_i = (a_i^{(k)})_{k=1}^{\infty}$. For every $k \in N$ let us denote

$$I(k) = \{i; i \in I,\ a_i^{(k)} \neq 0\}.$$

As $y_k = \{a_i^{(k)}\}_{i\in I} \in F = Z^{(I)}$ (direct sum), each set $I(k)$ is necessarily finite. If $m \in N$ then for $i \in I \setminus \bigcup_{k=1}^{m} I(k)$ we have $a_i^{(k)} = 0$ $(k=1,2,\ldots,m)$ and hence $\alpha_i \in p^m J_p$. Thus we conclude $\rho(y) \in J_p^{[I]}$ or $\operatorname{Im}\rho \subseteq J_p^{[I]}$. To prove the converse, consider $\{\alpha_i\}_{i\in I} \in J_p^{[I]}$ with $\alpha_i = (a_i^{(k)})_{k=1}^{\infty}$ satisfying (1),(2) and define the elements

$$y_k = \{a_i^{(k)}\}_{i\in I} \in Z^I \qquad (k=1,2,\ldots).$$

From the construction of $J_p^{[I]}$ it follows that for every $k \in N$ the set $\{i; i \in I, \alpha_i \notin p^k J_p\}$ is finite. But if $\alpha_i \in p^k J_p$ then $a_i^{(j)} = 0$ $(j=1,2,\ldots,k)$ and hence $y_k \in Z^{(I)} = F$. Now it is obvious that

$$y = (y_1 + pF,\ y_2 + p^2F, \ldots, y_k + p^kF, \ldots) \in \hat{F}_p$$

and $\rho(y) = \{\alpha_i\}_{i\in I}$. Therefore, ρ represents an isomorphism $\hat{F}_p \cong J_p^{[I]}$ and the proof is finished.

<u>Corollary 1</u>. If n is a cardinal and $F = Z^{(n)}$ then $\hat{F}_p \cong J_p^{[n]}$.

<u>Lemma 2</u>. Let A be a torsion free group and let n denote the rank of A/pA. Then $\hat{A}_p \cong J_p^{[n]}$.

Proof. If B is any p-basic subgroup of A then the sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

is p-pure exact with p-divisible group A/B. By [2, Theorem 39.8 and Exercise 39.6] we get the exact sequence

$$0 \longrightarrow \hat{B}_p \longrightarrow \hat{A}_p \longrightarrow (\widehat{A/B})_p \longrightarrow 0.$$

But the p-divisibility of A/B implies $(\widehat{A/B})_p = 0$ and we obtain $\hat{B}_p \cong \hat{A}_p$. As $B \cong Z^{(n)}$, the relations $\hat{A}_p \cong \hat{B}_p \cong J_p^{[n]}$ follow by Corollary 1.

<u>Lemma 3</u>. Let B be a p-pure subgroup of a torsion free group A such that the group A/B is p-divisible. Then for any group G the relation $\mathrm{Ext}(B,G)_{(p)} \neq 0$ implies $\mathrm{Ext}(A,G)_{(p)} \neq 0$.

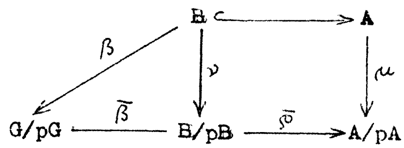
Proof. Consider any group G and denote by ε, μ, ν the natural homomorphisms

$$\varepsilon : G \longrightarrow G/pG, \qquad \mu : A \longrightarrow A/pA, \qquad \nu : B \longrightarrow B/pB.$$

Firstly we shall prove that each homomorphism $\beta : B \longrightarrow G/pG$ may be extended to a homomorphism $\alpha : A \longrightarrow G/pG$ in such a way that the diagram

$$(3) \qquad \begin{array}{ccc} B & \hookrightarrow & A \\ {\scriptstyle\beta}\downarrow & \swarrow{\scriptstyle\alpha} & \\ G/pG & & \end{array}$$

commutes. Indeed, let us define a homomorphism $\varphi : B \longrightarrow A/pA$ by setting $\varphi(b) = b + pA$ for every $b \in B$. Evidently, $\mathrm{Ker}\,\varphi = B \cap pA = pB$ (in view of the p-purity of B in A) and $\mathrm{Im}\,\varphi = (B + pA)/pA = A/pA$ (as the p-divisibility of A/B implies the equality $B + pA = A$). Thus φ induces a natural isomorphism $\bar{\varphi} : B/pB \longrightarrow A/pA$ defined by $\bar{\varphi}(b + pB) = b + pA$ for every $b \in B$. For given $\beta : B \longrightarrow G/pG$ it is $pB \subseteq \mathrm{Ker}\,\beta$ and hence, by the homomorphism theorem, there is a homomorphism $\bar{\beta} : B/pB \longrightarrow G/pG$ satisfying $\bar{\beta} \circ \nu = \beta$. Consequently, we get a commutative diagram of the form



and it suffices to put $\alpha = \bar{\beta} \circ \bar{\varphi}^{-1} \circ \mu$.

Let α, β be some homomorphisms of the commutative diagram (3). If there is $\psi \in \mathrm{Hom}(A,G)$ satisfying $\alpha = \varepsilon \circ \psi$ then β may be expressed in the form $\beta = \varepsilon \circ \varphi$ where φ is the restriction of ψ to the subgroup B. Now we are ready to prove our implication. By [1, Satz 3.2] the relation $\mathrm{Ext}(B,G)_{(p)} \neq 0$ implies the existence of a $\beta \in \mathrm{Hom}(B,G/pG)$ which cannot be expressed in the form $\beta = \varepsilon \circ \varphi$ with $\varphi \in \mathrm{Hom}(B,G)$. But then the corresponding $\alpha \in \mathrm{Hom}(A,G/pG)$ of the diagram (3) (its existence was just proved) has no expression of the form $\alpha = \varepsilon \circ \psi$ with $\psi \in \mathrm{Hom}(A,G)$. In view of the same [1, Satz 3.2] we conclude $\mathrm{Ext}/A,G)_{(p)} \neq 0$ and this completes the proof of Lemma.

Recall now that the group $Z^{\aleph_0}$ is usually denoted by P. If P(p) represents its subgroup of all $x = \{a_i\}_{i=1}^{\omega} \in P$ such that for every $n \in N$ the relation $p^n | a_i$ holds for almost all $i \in N$, then we have the inclusions

(4) $$Z^{(\aleph_0)} \subseteq P \subseteq J_p^{\aleph_0}, \quad Z^{(\aleph_0)} \subseteq P(p) \subseteq J_p^{[\aleph_0]}.$$

In what follows, a result of R. Baer [1] will appear very useful:

Lemma 4. If T is a torsion p-primary group then the following assertions are equivalent: 1) T is a direct sum of a divisible and a bounded groups; 2) $\mathrm{Ext}(P(p),T)_{(p)} = 0$; 3) $\mathrm{Ext}(P,T)_{(p)} = 0$.

Proof. See [1, Satz 4.1].

The next proposition is an analogous one.

Lemma 5. If T is a torsion p-primary group then the following assertions are equivalent: 1) T is a direct sum of a divisible and a bounded groups; 2) $\mathrm{Ext}(J_p^{[\aleph_0]}, T)_{(p)} = 0$; 3) $\mathrm{Ext}(J_p^{\aleph_0}, T)_{(p)} = 0$.

Proof. If T satisfies 1) then T is a cotorsion group and hence 2) and 3) hold; therefore, we have 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3). If T does not satisfy 1) then we shall prove that $\mathrm{Ext}(J_p^{[\aleph_0]}, T)_{(p)} \neq 0$ and also $\mathrm{Ext}(J_p^{\aleph_0}, T)_{(p)} \neq 0$. To this end we shall observe some properties of the groups P and P(p).

a) The subgroup P(p) (P resp.) is p-pure in the group $J_p^{\aleph_0}$. In fact, if $\{\alpha_i\}_{i=1}^{\infty} \in J_p^{\aleph_0}$ and $p^k \cdot \{\alpha_i\}_{i=1}^{\infty} \in P(p)$ ($\in P$ resp.) then for each $i \in N$ we have $p^k \cdot \alpha_i \in Z$ and hence $\alpha_i \in Q \cap J_p = Q_p$; but then the relation $p^k \cdot \alpha_i \in Z$ implies $\alpha_i \in Z$ ($i \in N$). Thus from $p^k \cdot \{\alpha_i\}_{i=1}^{\infty} \in P(p)$ ($\in P$ resp.) we conclude $\{\alpha_i\}_{i=1}^{\infty} \in P(p)$ ($\in P$ resp.).

b) The group $J_p^{\aleph_0}/P$ is p-divisible. To see this take any (canonically expressed) p-adic integer $\alpha = (a^{(k)})_{k=1}^{\infty}$ (compare with (1) and (2)); then $\alpha - a^{(k)} \in p^k J_p$ for every $k \in N$. But this means that $J_p^{\aleph_0} = P + p^k J_p^{\aleph_0}$ for every $k \in N$.

c) The group $J_p^{[\aleph_0]}/P(p)$ is p-divisible. Indeed, from the definition of the group $J_p^{[\aleph_0]}$ it follows that $J_p^{[\aleph_0]} = J_p^{(\aleph_0)} + p^k J_p^{[\aleph_0]}$ for every $k \in N$. But by the same argument as in b) we deduce that $Z^{(\aleph_0)} + p^k J_p^{(\aleph_0)} = J_p^{(\aleph_0)}$ and hence, in view of (4) we get

$$J_p^{[\aleph_0]} = Z^{(\aleph_0)} + p^k J_p^{[\aleph_0]} = P(p) + p^k J_p^{[\aleph_0]} .$$

This guarantees the p-divisibility of $J_p^{[\aleph_0]}/P(p)$.

Suppose now that the group T does not satisfy 1). Then by Baer theorem reformulated in Lemma 4 we get $\mathrm{Ext}(P(p),T)_{(p)} \neq$ $\neq 0$ and $\mathrm{Ext}(P,T)_{(p)} \neq 0$. In view of a) and c) and Lemma 3 we deduce $\mathrm{Ext}(J_p^{[\aleph_0]},T)_{(p)} \neq 0$. Analogously, the assertions a),b) amd the same Lemma 3 imply $\mathrm{Ext}(J_p^{\aleph_0},T)_{(p)} \neq 0$. This concludes the proof of our lemma.

As an immediate consequence we get

Lemma 6. Let n be any infinite cardinal and T a torsion p-primary group. Then the following assertions are equivalent: 1) T is a direct sum of a divisible and a bounded groups; 2) $\mathrm{Ext}(J_p^{[n]},T)_{(p)} = 0$; 3) $\mathrm{Ext}(J_p^{n},T)_{(p)} = 0$.

Proof. The implications 1) $\Longrightarrow$ 2) and 1) $\Longrightarrow$ 3) follow as in the proof of Lemma 5. If T does not satisfy 1) then it suffices to use Lemma 5 together with the fact that $J_p^{[\aleph_0]}$ ($J_p^{\aleph_0}$ resp.) is a direct summand of $J_p^{[n]}$ (J_p^{n} resp.).

The proof of the following theorem is based on some earlier author's results [4, 5]. Before we formulate it we recall that two groups G, H are said to be quasi-isomorphic (p-quasi-isomorphic resp.) if there are subgroups $U \subseteq G$, $V \subseteq H$ and a positive integer n such that $nG \subseteq U$, $nH \subseteq V$ ($p^nG \subseteq U$, $p^nH \subseteq V$ resp.) and $U \cong V$ (see [5]). The relation of the quasi-isomorphism (p-quasi-isomorphism resp.) will be written by $G \dot{\cong} H$ ($G \underset{p}{\dot{\cong}} H$ resp.).

Theorem. If A is a torsion free group and p a prime then the following assertions are equivalent: 1) The group A/pA is of finite rank; 2) $\hat{A}_p$ as Q_p^*-module is completely decomposable:, 3) the group $\hat{A}_p$ belongs to a Baer class Γ_α ; 4) $J_p \otimes_Z \hat{A}_p$ as Q_p^*-module is completely decomposable; 5) for every torsion group T it is $\mathrm{Ext}(\hat{A}_p,T)_{(p)} = 0$; 6) for every torsion group T and every group G the relation $G \underset{p}{\dot{\cong}} \hat{A}_p \oplus T$ implies the splitting of G; 7) for every torsion group T and every group G the relation $G \dot{\simeq} \hat{A}_p \oplus T$ implies the splitting of G.

Proof. The implication 1) $\Longrightarrow$ 2) follows by Lemma 2. If the Q_p^{κ}-module $\hat{A}_p$ is completely decomposable then it is a direct sum of the groups isomorphic either to J_p or to K_p where K_p is the additive group of the field of p-adic numbers. Then the group $\hat{A}_p$ belongs to a Baer class Γ_{α} and hence 2) $\Rightarrow$ $\Rightarrow$ 3). The implication 3) $\Longrightarrow$ 4) is proved in [4, Théorème 4*] and 4) $\Rightarrow$ 5) follows by [5, Proposition 5]. From [5, Proposition 3) we get the equivalence 5) $\Longleftrightarrow$ 6), the implication 7) $\Longrightarrow$ 6) is evident. Suppose now that 6) is fulfilled, take a torsion group T and consider any group G containing $\hat{A}_p \oplus T$ as a subgroup such that $G/(\hat{A}_p \oplus T)$ is bounded. Without loss of generality we may suppose that T is the maximal torsion subgroup of G. As $q\hat{A}_p = \hat{A}_p$ for every prime $q \neq p$, we deduce that $G/(\hat{A}_p \oplus T)$ is p-primary, therefore, $G \underset{p}{\dot{\simeq}} \hat{A}_p \oplus T$, and in view of 6) the group G splits. In fact, this proves the implication 6) $\Rightarrow$ 7). Finally, the implication 5) $\Longrightarrow$ 1) is a consequence of Lemma 6 and Lemma 7. The proof of Theorem is complete.

To conclude this remark we mention that [3, Corollary 4] concerns also the equivalence 1) $\Leftrightarrow$ 4). But the proof methods here and in [3] are fully different.

R e f e r e n c e s

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