Commentationes Mathematicae Universitatis Carolinae

Petr Simon

A compact Fréchet space whose square is not Fréchet

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 4, 749--753

Persistent URL: `http://dml.cz/dmlcz/106040`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,4 (1980)

A COMPACT FRÉCHET SPACE WHOSE SQUARE IS NOT FRÉCHET

Petr SIMON

Abstract: We shall prove in ZFC only that there exist two compact Hausdorff Fréchet spaces X_1, X_2 such that $X_1 \times X_2$ is not Fréchet.

Key words and phrases: MAD family, Fréchet space.

Classification: 54D30, 54D55

In 1977, E. Michael raised the question whether there existed two spaces X_1, X_2, both compact Hausdorff and Fréchet, the product of which was not Fréchet ([Mi], Problem 3). A topological space X is Fréchet if for each non-void $M \subseteq X$ and for each $x \in \bar{M}$ there is a sequence $\{x_n : n \in \omega\} \subseteq M$ converging to x. Assuming various set-theoretical axioms, V.I. Malychin [Ma], R.C. Olson [O], T.K. Boehme and M. Rosenfeld [BR] gave examples of such spaces. All those examples are twin brothers - they are Franklin compacta (the definition is given below) constructed from some suitable almost disjoint family on N; our example is yet another one of the same nature. The heart and soul of all the constructions mentioned lies in the existence of a "well-behaving" maximal almost disjoint system. We shall show that the MAD family needed really exists.

Let us recollect some necessary notions and facts. N will denote the set of all natural numbers (and, if considered as a topological space, its topology is discrete). An almost disjoint family (abbr. AD) on a set X is a collection $\mathcal{P} \subseteq [X]^{\omega}$ such that $P \cap P'$ is finite for any two distinct members $P, P' \in \mathcal{P}$. A maximal almost disjoint family (abbr. MAD) on X is an AD family on X properly contained in no AD family on X.

Let $\mathcal{P}$ be AD on N, let $X \in [N]^{\omega}$. Denote $X \wedge\wedge \mathcal{P} = \{P \cap X : P \in \mathcal{P}$ and $|P \cap X| = \omega\}$. Let $\mathcal{J}(\mathcal{P}) = \{X \in [N]^{\omega} : X \wedge\wedge \mathcal{P}$ is finite$\}$, $\mathcal{J}^{+}(\mathcal{P}) = [N]^{\omega} - \mathcal{J}(\mathcal{P}) = \{X \in [N]^{\omega} : X \wedge\wedge \mathcal{P}$ is infinite$\}$, $\mathcal{M}(\mathcal{P}) = \{X \in [N]^{\omega} : X \wedge\wedge \mathcal{P}$ is MAD on $X\}$.

For $A \subseteq N$, denote as usual $A^{*} = \mathrm{cl}A - A$, where the closure of A is taken in βN, the Čech-Stone compactification of integers. Then for $X \in [N]^{\omega}$, $\mathcal{P}$ AD on N, the set X belongs to $\mathcal{M}(\mathcal{P})$ if and only if $X^{*} \subseteq \mathrm{cl} \bigcup \{P^{*} : P \in \mathcal{P}\}$.

Let $\mathcal{P}$ be AD family on N. The Franklin compact $\mathcal{F}(\mathcal{P})$ is a topological space whose underlying set is $N \cup \mathcal{P} \cup \{\infty\}$ and whose topology is given as follows: N is a set of isolated points, a basic open neighborhood of a point $P \in \mathcal{P}$ is $\{P\} \cup$ $\cup$ cofinite subset of P, ∞ is a point distinct from all $n \in N$ and all $P \in \mathcal{P}$, which compactifies the space $N \cup \mathcal{P}$. Equivalently, $\mathcal{F}(\mathcal{P})$ is a quotient space of βN modulo the equivalence $x \sim x'$ iff $x, x' \in N^{*}$ and either $\{x, x'\} \subseteq P^{*}$ for some $P \in \mathcal{P}$ or $\{x, x'\} \cap P^{*} = \emptyset$ for all $P \in \mathcal{P}$. Clearly $\mathcal{F}(\mathcal{P})$ is a compact Hausdorff space.

The crucial properties of Franklin compacta were stated by V.I. Malychin in [Ma]:

(a) $\mathcal{F}(\mathcal{P})$ is a Fréchet space iff $N^{*} - \bigcup \{P^{*} : P \in \mathcal{P}\}$ is a regular closed set in N^{*}, equivalently, iff $\mathcal{M}(\mathcal{P}) \subseteq \mathcal{J}(\mathcal{P})$.

(b) If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is AD on N and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, then the product $\mathcal{F}(\mathcal{P}_1) \times \mathcal{F}(\mathcal{P}_2)$ is not Fréchet iff $N^* - \bigcup\{P^*: :P \in \mathcal{P}\}$ is not regular closed in N^*, equivalently, iff $\mathcal{M}(\mathcal{P}) \cap \mathcal{J}^+(\mathcal{P}) \neq \emptyset$.

(c) In particular, $\mathcal{M}(\mathcal{P}) \cap \mathcal{J}^+(\mathcal{P}) \neq \emptyset$ if $\mathcal{P}$ is an infinite MAD system on N, hence it suffices to show the following:

Theorem. There is a MAD family $\mathcal{P}$ on N and its partition $\mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}$ such that $\mathcal{M}(\mathcal{P}_i) \subseteq \mathcal{J}(\mathcal{P}_i)$ for i = 0,1.

Indeed, by (a), $\mathcal{F}(\mathcal{P}_0)$ as well as $\mathcal{F}(\mathcal{P}_1)$ is Fréchet, but by (b) and (c), $\mathcal{F}(\mathcal{P}_0) \times \mathcal{F}(\mathcal{P}_1)$ fails to be.

Before giving a proof, let us state and prove a lemma, due to J. Dočkálková:

Lemma [D]. Let $\mathcal{P}$ be an infinite MAD family on N, $\{X_0 \supseteq \supseteq X_1 \supseteq X_2 \supseteq \dots\}$ a countable subset of $\mathcal{J}^+(\mathcal{P})$. Then there is a set $Y \in \mathcal{J}^+(\mathcal{P})$ such that for each $n \in \omega$, $Y - X_n$ is finite.

Proof. Choose $y(0,n) \in X_n$ for each $n \geq 0$, $y(0,n+1) > y(0,n)$. The set $Y(0) = \{y(0,n): n \geq 0\}$ is infinite and $\mathcal{P}$ is MAD, hence there is some $P_0 \in \mathcal{P}$ with $P_0 \cap Y(0)$ infinite. Set $X(1)_n = X_n - - P_0$. Since $X_n \in \mathcal{J}^+(\mathcal{P})$, the set $X(1)_n$ belongs to $\mathcal{J}^+(\mathcal{P})$, too.

Choose $y(1,n) \in X(1)_n$ for each $n \geq 1$, $y(1,n+1) > y(1,n)$. The set $Y(1) = \{y(1,n): n \geq 1\}$ is infinite and $\mathcal{P}$ is MAD, hence there is some $P_1 \in \mathcal{P}$ with $P_1 \cap Y(1)$ infinite. Clearly $P_1 \neq P_0$ because $P_0 \cap X(1)_n = \emptyset$ for all n. Proceeding by an induction ($y(k,n) \in X(k)_n$ are chosen for $n \geq k$ only), we obtain the set $Y = \bigcup\{Y(k) \cap P_k : k \in \omega\}$, which has the desired properties. □

Proof of the theorem. Suppose the theorem to be false, i.e.

(*) for each MAD family $\mathcal{P}$ on a countably infinite set and for each partition $\mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}$ there is an $i \in \{0,1\}$ and a set $X_i \in \mathcal{J}^+(\mathcal{P}_i) \cap \mathcal{M}(\mathcal{P}_i)$.

Let $\mathcal{P}$ be a MAD family of size continuum on N. Enumerate $\mathcal{P}$ as $\mathcal{P} = \{P_f : f \in {}^{\omega}2\}$. Let $\mathcal{P}_{n,i} = \{P_f : f(n) = i\}$ for $n \in \omega$, $i \in \{0,1\}$. Clearly for each $n \in \omega$, $\mathcal{P}_{n,0} \cup \mathcal{P}_{n,1} = \mathcal{P}$, $\mathcal{P}_{n,0} \cap \mathcal{P}_{n,1} = \emptyset$.

Induction. $n = 0$: By (*), there is some $i_0 \in \{0,1\}$ and a set $X_0 \in \mathcal{J}^+(\mathcal{P}_{0,i_0}) \cap \mathcal{M}(\mathcal{P}_{0,i_0})$. Thus $X_0 \in \mathcal{J}^+(\mathcal{P})$.

$n = 1$: $X_0 \wedge\wedge \mathcal{P}$ is a MAD family on X_0 and $\{X_0 \wedge\wedge \mathcal{P}_{1,0}, X_0 \wedge\wedge \mathcal{P}_{1,1}\}$ is its partition. By (*), there is some $i_1 \in \{0,1\}$ and a set $X_1 \in \mathcal{J}^+(X_0 \wedge\wedge \mathcal{P}_{1,i_1}) \cap \mathcal{M}(X_1 \wedge\wedge \mathcal{P}_{1,i_1})$. Clearly $X_1 \in \mathcal{J}^+(\mathcal{P})$.

$n = 2$: $X_1 \wedge\wedge \mathcal{P}$ is a MAD family on X_1 and $\{X_1 \wedge\wedge \mathcal{P}_{2,0}, X_1 \wedge\wedge \mathcal{P}_{2,1}\}$ is its partition. By (*), there is some $i_2 \in \{0,1\}$ and ... it is obvious how to proceed further on.

At the end we obtain a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ and a sequence $\{i_n : n \in \omega\}$ of zeros and ones such that $X_n \in \mathcal{J}^+(\mathcal{P}_{n,i_n}) \cap \mathcal{M}(\mathcal{P}_{n,i_n})$. Let $f \in {}^{\omega}\{0,1\}$ be defined by $f(n) = i_n$, let $Y \in \mathcal{J}^+(\mathcal{P})$ be the set the existence of which is guaranteed by the lemma: $Y - X_n$ is finite for each $n \in \omega$. Since $Y \in \mathcal{J}^+(\mathcal{P})$, we have $|Y \cap P_g| = \omega$ for infinitely many g's from ${}^{\omega}\{0,1\}$, pick one such g distinct from f. For some $n \in \omega$, $f(n) \neq g(n)$, fix this n.

From $|Y - X_n| < \omega$ and $|Y \cap P_g| = \omega$ follows that $|X_n \cap P_g| = \omega$. Now $P_g \notin \mathcal{P}_{n,f(n)}$, hence $|P_g \cap P| < \omega$ for each $P \in \mathcal{P}_{n,f(n)}$ and $X_n \cap P_g$ is infinite, yet $X_n \wedge\wedge \mathcal{P}_{n,f(n)}$ is MAD on X_n - a contradiction. □

Remark. A more detailed examination of the proof just given shows that a bit more is valid, namely:

For each infinite MAD family $\mathcal{P}$ on N there is some $X \in \in \mathcal{J}^+(\mathcal{P})$ such that $X \wedge\wedge \mathcal{P}$ is a MAD family on X having the property stated in Theorem.

R e f e r e n c e s

[BR] T.K. BOEHME, M. ROSENFELD: An example of two compact Fréchet Hausdorff spaces, whose product is not Fréchet, J. London Math. Soc. 8(1974), 339-344.

[D] J. DOČKÁLKOVÁ: Almost disjoint refinement of families of subsets of natural and real numbers (in Czech), Rigorosní práce, ČKD Praha, závod Polovodiče, Praha 1980.

[Ma] V.I. MALYCHIN: O sekvencial'nych i Freše-Urysona bikompaktach, Vestnik Moskov. Univ. 5(1976), 42-47.

[Mi] E. MICHAEL: A quintuple quotient quest, Gen. Top. and Appl. 2(1972), 91-138.

[Mt] A.R.D. MATHIAS: Happy families, Ann. Math. Logic 12(1977) 59-111.

[O] R.C. OLSON: Bi-quotient maps, countable bi-sequential spaces and related topics, Gen. Top. and Appl. 4(1974), 1-28.

Matematický ústav
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 15.9. 1980)