Commentationes Mathematicae Universitatis Carolinae

Ladislav Bican

A note on the splitting length of a finite direct sum of mixed Abelian groups of rank one

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 4, 739--747

Persistent URL: `http://dml.cz/dmlcz/106039`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,4 (1980)

A NOTE ON THE SPLITTING LENGTH OF A FINITE DIRECT SUM OF MIXED ABELIAN GROUPS OF RANK ONE

Ladislav BICAN

Abstract: The purpose of this note is to show that the splitting length of a finite direct sum A of mixed abelian groups of rank one does not depend on the splitting lengths of the summands provided that the rank of A is greater than 1 and at least one of the *summands is non-splitting. More precisely, it is shown that the splitting length of a direct sum of* mixed abelian *groups* $A_1, A_2, \dots, A_m$ *of rank one with the splitting* lengths $k_1 \le k_2 \le \dots \le k_m$, $m \ge 2$, $k_m \ge 2$, *can take an arbitrary va*lue from the set $\{k_m, k_m+1, \dots, \infty\}$.

Key words: Splitting length, p-height sequence.

Classification: Primary 20K25

Irwin, Khabbaz and Rayna [8] have studied the splitting properties of the tensor product of mixed abelian groups. They defined the splitting length of a mixed group G as the infimum of the set of all positive integers n such that the n-th tesor $G^n = G \otimes G \otimes \overset{n\text{-times}}{\dots} \otimes G$ splits and they constructed a mixed group of rank one having the splitting length n for every positive integer n. In my previous paper [3] I have characterized the mixed abelian groups of rank one having the splitting length n and in [4] I have characterized all pairs A, B of mixed abelian groups of rank one having the property that the tensor product $A \otimes B$ splits. In this note we are going to prove the following result.

Theorem. Let $k_1, k_2, \dots, k_m$, $m \geq 2$, be arbitrary positive integers, not all equal to 1. Then for each ℓ, $\max\{k_1, k_2, \dots \dots, k_m\} \leq \ell \leq \infty$, there exist abelian groups $A_1, A_2, \dots, A_m$ such that each A_i has the splitting length k_i, $i=1,2,\dots,m$, and the direct sum $A = A_1 \oplus A_2 \oplus \dots \oplus A_m$ has the splitting length ℓ.

Thus, the splitting length of a finite direct sum A of mixed abelian groups of rank one does not depend on the splitting lengths of the summands provided that the rank of A is at least 2 and at least one of the summands is non-splitting.

By the word "group" we shall always mean an additively written abelian group. As in [1], we use the notions "characteristic" and "type" in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^A(a)$, $\tau^A(a)$ and $\hat{\tau}^A(a)$ denote respectively the p-height, the characteristic and the type of the element a in the group A. π will denote the set of all primes. If T is a torsion group, then T_p is the p-primary component of T and similarly, if $\pi' \subseteq \pi$ then $T_{\pi'}$ is defined by $T_{\pi'} = \sum_{p \in \pi'}^{\oplus} T_p$. If $\pi' \subseteq \pi$ and if A is a mixed group with the torsion part T(A), $T(A)_{\pi'} = 0$, then for each subset $S \subseteq A$ the symbol $\langle S \rangle_{\pi'}^{A}$ denotes the π'-pure closure of S in A, the existence of which is easily seen.

For a mixed group A with the torsion part T(A) we denote by $\bar{A}$ the factor-group A/T(A) and for $a \in A$, $\bar{a}$ is the element a + T(A) of $\bar{A}$. The symbol $|a|$ means the order of the element $a \in A$. The rank of a mixed group A is that of $\bar{A}$. The set of all positive integers is denoted by $\underline{N}$, $\underline{N}_0 = \underline{N} \cup \{0\}$. Other notation will be essentially the same as in [5].

It has been proved in [1; Theorem 2] that a mixed group A of rank one splits if and only if each element $a \in A \setminus T(A)$ has a non-zero multiple ma such that $\hat{\tau}^A(ma) = \hat{\tau}^{\bar{A}}(\bar{a})$ and ma has a p-sequence whenever $h_p^{\bar{A}}(\bar{a}) = \infty$ (i.e. there exist elements $h_0^{(p)} = ma$, $h_1^{(p)}, \dots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n=0,1,\dots$). Recall [3], that the p-height sequence of an element $a \in A$ is the double sequence $\{k_i, \ell_i\}_{i=0}^{\infty}$ of elements of $\underline{N}_0 \cup \{\infty\}$ defined inductively in the following way: Put $k_1 = k_0 = \ell_0 = 0$ and $\ell_1 = h_p^A(a)$. If k_i, ℓ_i are defined and either $h_p^A(p^{k_i}a) = \ell_i = \infty$, or $\ell_i < \infty$ and $h_p^A(p^{k_i+k}a) = \ell_i + k$ for all $k \in N$ then put $k_{i+1} = k_i$ and $\ell_{i+1} = \ell_i$. If $\ell_i < \infty$ and there are $k \in \underline{N}$ with $h_p^A(p^{k_i+k}a) > \ell_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^A(p^{k_{i+1}}a) = \ell_{i+1} > \ell_i + k_{i+1} - k_i$.

For the sake of simplicity we shall use the notation $a^r = a \otimes a \otimes \dots \otimes a \in A^r$, $r \in \underline{N}$. Moreover, the symbols $A^r \otimes B^0$ and $a^r \otimes b^0$, $r \in \underline{N}$, will simply denote A^r and a^r, respectively.

If $\pi' \subseteq \pi$ and $\ell \in \underline{N}_0$ then we shall say that a torsion-free group A of rank one is of the type $(\pi';\ell)$ if it contains an element a such that $h_p^A(a) = \ell$ for each $p \in \pi'$ and $h_p^A(a) = 0$ for each $p \in \pi \setminus \pi'$. Further, if $\pi' \subseteq \pi$ and $k, \ell, m \in \underline{N}_0$, $m > k + \ell$, then we shall say that a mixed group A of rank one is of the type $(\pi';k,\ell,m)$ if $T(A)_{\pi \setminus \pi'} = 0$ and $A \setminus T(A)$ contains an element a such that for each prime $p \in \pi'$ the p-height sequence of a in A is $\{k_i, \ell_i\}_{i=0}^{\infty}$, where $k_2 = k_3 = \dots = k$, $\ell_1 = \ell$, $\ell_2 = \ell_3 = \dots = m$ and for each prime $p \in \pi \setminus \pi'$ the p-height sequence of a in A is $\{k_i, \ell_i\}_{i=0}^{\infty}$, where $k_0 = k_1 = \dots = \ell_0 = \ell_1 = \dots = 0$.

We start our investigations with some preliminary lemmas.

<u>Lemma 1</u>. If $\pi' \subseteq \pi$ and $\ell \in \underline{N}_0$ are arbitrary then there

exists a torsionfree group A of rank one and of the type $(\pi'; \ell)$.

Proof: Let $\pi' = \{p_i \mid i \in I\}$, $U = \langle \tilde{a} \rangle \oplus \sum_{i \in I}^{\oplus} \langle a_i \rangle$ be a free group and $V = \langle \tilde{a} - p_i^{\ell} a_i \mid i \in I \rangle$ be its subgroup. It is an easy exercise to show that the factor-group $A = U/V$ is torsionfree of rank one and the element $a = \tilde{a} + V$ has the desired properties.

Lemma 2. Let $\pi' \subseteq \pi$ be an arbitrary set of primes. If $k, \ell, m \in \underline{N}$, $m > k + \ell$, then there exist a mixed group A of rank one and of the type $(\pi'; k, \ell, m)$.

Proof: Let $\pi' = \{p_i \mid i \in I\}$, $U = \langle \tilde{a} \rangle \oplus \sum_{i \in I}^{\oplus} (\langle a_i^{(1)} \rangle \oplus \oplus \langle a_i^{(2)} \rangle)$ be a free group and $V = \langle \tilde{a} - p_i^{\ell} a_i^{(1)}, p_i^k \tilde{a} - p_i^m a_i^{(2)} \mid$ $\mid i \in I \rangle$ be its subgroup. Obviously, the factor-group $A = U/V$ is of rank one and we are going to show that the element $a = \tilde{a} + V$ has the desired properties.

If the equation $p^s x = p^r a$ is solvable in A then $p^r \tilde{a} =$ $= p^s(\lambda \tilde{a} + \sum_{i \in I} \lambda_i a_i^{(1)} + \sum_{i \in I} \mu_i a_i^{(2)}) + \sum_{i \in I} \varrho_i (\tilde{a} - p_i^{\ell} a_i^{(1)}) +$ $+ \sum_{i \in I} \sigma_i (p_i^k \tilde{a} - p_i^m a_i^{(2)})$ (all the sums have only a finite number of non-zero terms) and consequently

(1) $p^r = p^s \lambda + \sum_{i \in I} \varrho_i + \sum_{i \in I} p_i^k \sigma_i$,

(2) $0 = p^s \lambda_i - p_i^{\ell} \varrho_i$, $i \in I$,

(3) $0 = p^s \mu_i - p_i^m \sigma_i$, $i \in I$.

If $p \notin \pi'$ then $p^s \mid \varrho_i$, $p^s \mid \sigma_i$, $i \in I$, by (2) and (3), hence $p^s \mid p^r$ by (1) and $h_p^A(p^r a) = r$.

Assume now that $p = p_j$ for some $j \in I$. If $r \in \{0, 1, \dots, k-1\}$ then $p_j^r \tilde{a} = p_j^{r+\ell} a_j^{(1)} + p_j^r(\tilde{a} - p_j^{\ell} a_i^{(1)})$ and so $h_p^A(p^r a) \geq r + \ell$. If $s > r + \ell$ then (2) and (3) yield $\varrho_i \equiv 0 \pmod{p^s}$, $\sigma_i \equiv 0$

(mod p^s), $j \neq i \in I$, and $\wp_j \equiv 0$ (mod $p^{s-\ell}$). Hence, by (1), $p^r \equiv p^k \sigma'_j$ (mod $p^{s-\ell}$) and so $1 \equiv p^{k-r} \sigma'_j$ (mod $p^{s-\ell-r}$), a contradiction. Thus $h_p^A(p^r a) = r + \ell$ for each $r \in \{0,1,\dots,k-1\}$.

Suppose now that $r \geq k$. Then obviously $p_j^r \tilde{a} = p_j^{m+r-k} a_j^{(2)} + p_j^{r-k}(p_j^k \tilde{a} - p_j^m a_j^{(2)})$ and so $h_p^A(p^r a) \geq m + r - k$. If $s > m + r - k$ then (2) and (3) yield $\wp_i \equiv 0$ (mod p^s), $\sigma_i \equiv 0$ (mod p^s), $j \neq i \in I$, and $\wp_j = 0$ (mod $p^{s-\ell}$), $p^{s-m}\sigma'_j = \sigma_j$ for a suitable integer σ'_j. Hence, by (1), $p^r \equiv p^{s-m+k}\sigma'_j$ (mod $p^{s-\ell}$) and so $1 \equiv p^{s-m-r+k}\sigma'_j$ (mod $p^{s-\ell-r}$), a contradiction. Thus $h_p^A(p^r a) = m + r - k$ for each $r \geq k$ and the proof is complete.

<u>Lemma 3</u>. Let A be a mixed group of rank one. If $\pi' \subseteq \pi$ is infinite and if A is of the type $(\pi';k-1,1,k+1+m)$, $k \geq 2$, $m \in \underline{N}_0$, then A has the splitting length k and for each $r \in \{1,2,\dots,k-1\}$ the tensor power A^r is of the type $(\pi';k-r,r,r(m+1)+k)$.

Proof: Obviously, $(k-1)(k+1+m-(k-1))-(k-1) = (k-1)(m+1) > 0$ and A has the splitting length k by [3; Theorem].

If $a \in A \setminus T(A)$ is an element having the properties stated in the definition of the group of the type $(\pi',k-1,1,k+1+m)$ then the assumption $T(A)_{\pi \setminus \pi'} = 0$ obviously yields $h_p^{A^r}(p^s a^r) = s$ for each prime $p \in \pi \setminus \pi'$.

Assume now that $p = p_j$ for some $j \in I$. It is easy to see that for each $a' \in A$ the p-heights of the elements $a', a' + T_{\pi \setminus \{p\}}$ in the corresponding groups are the same and from this it easily follows that we can restrict ourselves to the case of T(A) p-primary. If $pa_1 = a$, $p^{k+1+m}a_2 = p^{k-1}a$ and $t = p^{m+1}a_2 - a_1$ then by [4; Lemma 8] and [3; Lemma 8] the group A decomposes into $A = \langle t \rangle \oplus V \oplus \langle a_2 \rangle^A_{\pi \setminus \{p\}}$, where $\langle t \rangle \oplus V = T(A)$. Moreover,

$a = p^{m+2}a_2 - pt$ and so $h_p^{A^r}(a^r) = r$. Finally, $p^{k-r}a^r =$
$= p^{k-1}(a \otimes a_1^{r-1}) = p^{k+1+m}(a_2 \otimes a_1^{r-1}) = \dots = p^{(m+1)r+k}a_2^r$, from which the assertion follows easily.

Lemma 4. Let $\pi' \subseteq \pi$ be infinite and A, B be mixed groups of rank one and of the type $(\pi';k-1,1,k+m+2)$, $(\pi';(m-1)(\ell-1),\ell-1,m\ell)$, $k \geq 2$, $\ell \geq m \geq 3$, respectively. Then $A^{\ell-2} \otimes B$ does not split. The same holds if A is a torsionfree group of rank one and of the type $(\pi';m-1)$.

Proof: If $\ell - 2 < k$ then $A^{\ell-2}$ is of the type $(\pi';k-\ell+2,\ell-2,(\ell-2)(m-2)+k)$ by Lemma 3 and if $\ell - 2 \geq k$ then $A^{\ell-2}$ splits and its torsionfree direct summand is of the type $(\pi';(m-1)(\ell-2))$. In both cases we have $(\ell-2)(m-2)+k - (k-\ell+2) - (m-1)(\ell-1) = (m-1)(\ell-2) - (m-1)(\ell-1) = -(m-1) < 0$ and $A^{\ell-2} \otimes B$ does not split by [4; Theorem] (or [4; Corollary 3]). The rest is similar.

Lemma 5. Let $A_1,A_2,\dots,A_m$ be mixed abelian groups, $m \in N$. Then $(A_1 \oplus A_2 \oplus \dots \oplus A_m)^{\ell}$ splits if and only if $A_1^{r_1} \otimes A_2^{r_2} \otimes \dots \otimes A_m^{r_m}$ splits for all $r_1,r_2,\dots,r_m \in \underline{N}_0$ with $\sum_{i=1}^{m} r_i = \ell$.

Proof: The assertion follows easily from the simple fact that $A \oplus B$ splits if and only if both A and B are splitting.

Proof of Theorem: With respect to Lemma 5 we can suppose that $k_1 \leq k_2 \leq \dots \leq k_m < \infty$. Now we shall divide the proof into several cases.

I. $\ell < \infty$.

1. Let $k_m \geq 3$ and let $j \in \{0,1,\dots,m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} \leq \dots \leq k_m$. For each $i \in \{1,2,\dots,j\}$ let A_i be a torsionfree group of rank one of the type $(\pi;k_m-1)$, for each $i \in \{j+1,\dots,m-1\}$ let A_i be a mixed group of rank one

of the type (π ;$k_i-1,1,k_i+k_m-2$) and let A_m be a mixed group of rank one of the type (π ;$(k_m-1)(\ell-1),\ell-1,k_m\ell$). The groups A_i, $i=1,2,\dots,m$, have the splitting length k_i by Lemma 3, the group $A_{m-1}^{\ell-2}\otimes A_m$ does not split by Lemma 4, so that with respect to Lemma 5 it remains to show that $A_1^{r_1}\otimes A_2^{r_2}\otimes\dots\otimes A_m^{r_m}$ splits whenever $r_1,r_2,\dots,r_m\in \underline{N}_0$ and $\sum_{i=1}^{m} r_i=\ell$. By [1; Theorem 2] it suffices to show that

$$h_p^{A_1^{r_1}\otimes\dots\otimes A_m^{r_m}}(a_1^{r_1}\otimes a_2^{r_2}\otimes\dots\otimes a_m^{r_m})=\ell(k_m+r_m-1)\ (a_1,a_2,\dots,a_m$$

are the elements having the properties stated in the definition of the groups of the corresponding types). For each $p\in\pi$ we have $p^{k_m-1}a_i^{(1)}=a_i$, $a_i^{(1)}\in A_i$, $i\in\{1,2,\dots,j\}$, $pa_i^{(1)}=a_i$, $p^{k_i+k_m-2}a_i^{(2)}=p^{k_i-1}a_i$, $a_i^{(1)}$, $a_i^{(2)}\in A_i$, $i\in\{j+1,\dots,m-1\}$, and $p^{\ell-1}a_m^{(1)}=a_m$, $p^{k_m\ell}a_m^{(2)}=p^{(k_m-1)(\ell-1)}a_m$, $a_m^{(1)},a_m^{(2)}\in A_m$. Now let k be the first integer with $r_k>0$. If $k=m$ then $a_m^{\ell}=p^{\ell(\ell-1)}(a_m^{(1)})^{\ell}$, $p^{k_m(\ell-1)}a_m^{(1)}=p^{k_m\ell}a_m^{(2)}$ and the induction yields $a_m^{\ell}=$ $=p^{\ell(\ell+k_m-1)}(a_m^{(2)})^{\ell}$ owing to the fact that $\ell(\ell-1)-k_m(\ell-1)=$ $=(\ell-1)(\ell-k_m)\geq 0$. Now if $k\in\{j+1,\dots,m-1\}$ then $p^{k_i}a_i^{(1)}=$ $=p^{k_i+k_m-2}a_i^{(2)}$, $i\in\{k,k+1,\dots,m-1\}$. If we put $\alpha=\ell+r_m(\ell-2)$ and $\beta=\alpha+(k_m-2)(\ell-r_m)$ then $\alpha\geq\ell\geq k_i$, $i\in\{k,k+1,\dots,m-1\}$, and $a_k^{r_k}\otimes\dots\otimes a_m^{r_m}=p^{\alpha}((a_k^{(1)})^{r_k}\otimes\dots\otimes(a_m^{(1)})^{r_m})=$ $=p^{\beta}((a_k^{(2)})^{r_k}\otimes\dots\otimes(a_{m-1}^{(2)})^{r_{m-1}}\otimes(a_m^{(1)})^{r_m})$. For $r_m=0$ we are ready and for $r_m>0$ the inequality $\beta-k_m(\ell-1)=$ $=(\ell-k_m)(r_m-1)\geq 0$ yields $a_k^{r_k}\otimes\dots\otimes a_m^{r_m}=p^{\ell(k_m+r_m-1)}$ $((a_k^{(2)})^{r_k}\otimes\dots\otimes(a_m^{(2)})^{r_m})$. Finally, if $k\in\{1,2,\dots,j\}$ then

$\alpha = \ell + r_m(\ell - 2) + (k_m - 2)\sum_{i=k}^{j} r_i \geq \ell \geq k_i$, $i \in \{j+1,\dots,m-1\}$, $\beta = \alpha + (k_m - 2)\sum_{i=k}^{m-1} r_i$, $\beta - k_m(\ell - 1) = (\ell - k_m)(r_m - 1)$ and the assertion follows as in the preceding case.

2. $k_m = 2$.

a) Let $\ell \geq 3$ and let $j \in \{0,1,\dots,m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} = \dots = k_m = 2$. For each $i \in \{1,2,\dots,j\}$ let A_i be a torsionfree group of rank one of the type $(\pi\ ;2)$, for each $i \in \{j+1,\dots,m-1\}$ let A_i be a mixed group of rank one of the type $(\pi\ ;1,1,3)$ and let A_m be a mixed group of rank one of the type $(\pi\ ;2(\ell-1),\ \ell-1,\ 4(\ell-1))$. The groups A_i, $i=1,2,\dots,m$, have the splitting length k_i by Lemma 3 and the group $A_{m-1}^{\ell-2} \otimes A_m$ does not split by Lemma 4. The proof of the splitting of A is similar to that in 1.

b) Let $\ell = 2$ and let $j \in \{0,1,\dots,m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} = \dots = k_m = 2$. For each $i \in \{1,2,\dots,j\}$ let A_i be a torsionfree group of rank one of the type $(\pi\ ;1)$ and for each $i \in \{j+1,\dots,m\}$ let A_i be a mixed group of rank one of the type $(\pi\ ;1,1,3)$. The groups A_i, $i=1,2,\dots,m$ have the splitting length k_i and the splitting length of A is obviously 2.

II. $\ell = \infty$.

Let p be a prime and $j \in \{0,1,\dots,m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} \leq \dots \leq k_m$. For each $i \in \{1,2,\dots,j\}$ let $A_i = Z$ (the group of integers), for each $i \in \{j+1,\dots,m-1\}$ let A_i be a mixed group of rank one of the type $(\pi \setminus \{p\};\ k_i-1,1,k_i+1)$ and let A_m be the group generated by the elements $a_0, a_1, \dots$ with respect to the relations $p^{(k_m-1)i} a_i = p^{(k_m-2)i} a_0$. The groups A_i, $i=1,2,\dots,m-1$, have the splitting length k_i by Lemma 3 and the group A_m has the splitting length k_m by [3; Example] (see also [8]). However, for each $\ell > 1$ the group $A_{m-1}^{\ell-1}$ is p-reduced, no non-zero element from A_m has a p-sequence and hence the group $A_{m-1}^{\ell-1} \otimes A_m$ does not split by [4; Theorem]. Thus the group A is of infinite splitting length and the proof is complete.

References

[1] L. BICAN: Mixed abelian groups of torsionfree rank one, Czech. Math. J. 20(1970), 232-242.

[2] L. BICAN: A note on mixed abelian groups, Czech. Math. J. 21(1971), 413-417.

[3] L. BICAN: The splitting length of mixed abelian groups of rank one, Czech. Math. J. 27(1977), 144-154.

[4] L. BICAN: The splitting of the tensor product of two mixed abelian groups of rank one (to appear).

[5] L. FUCHS: Abelian groups, Budapest, 1958.

[6] L. FUCHS: Infinite abelian groups I, Academic Press, New York and London, 1970.

[7] L. FUCHS: Infinite abelian groups II, Academic Press, New York and London, 1973.

[8] I.M. IRWIN, S.A. KHABBAZ, G. RAYNA: The role of the tensor product in the splitting of abelian groups, J. Algebra 14(1970), 423-442.

Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 2.7. 1980)