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DOES SP K ⊇ PS K IMPLY AXIOM OF CHOICE?

H. ANDRÉKA and I. NÉMETI

Abstract: Problem 28 in Grätzer [2] asks what the semigroup generated by the operators **I**, **H**, **S**, **P** etc. (on classes of algebras) is like without the Axiom of Choice (AC). The present paper contains partial answers to this question, e.g. **SIP**, **HSP** are closure operators without AC, but the AC is provable from any one of the following assumptions: **IP** is a closure operator, **HP** is a closure operator, **SP** ⊇ **PS**. Some questions are formulated at the end of the paper, e.g. whether **IPIP** is a closure operator without AC or not.

Key words: Universal algebra, Axiom of choice, operators on classes of algebras, reduced products, direct products, varieties, quasivarieties.

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Secondary 08C15, 08C99

Notations. Let K be any class of similar algebras. **P** K denotes the class of all algebras isomorphic to direct products of elements of K (see [3]). Similarly, $\mathbf{P}^r$ K denotes the class of all algebras isomorphic to reduced products of elements of K. $\mathbf{P}^g$ K denotes the class of all algebras which are direct products of elements of K (see [2] p. 152). **S** K is the class of all subalgebras of elements of K, and **I** K is the class of all algebras isomorphic to elements of K.

Here we assume that the universe of an algebra is nonempty. Therefore without AC, direct products of algebras need not exist.

Therefore $\mathbf{P}^g$ K is defined so that whenever a product of elements of K exists (i.e. is nonempty), this product is an element of $\mathbf{P}^g$ K. Similarly for $\mathbf{P}$, $\mathbf{P}^r$. The precise definition goes as follows:

$\mathbf{P}^g K \stackrel{df}{=} \{ \underline{\underline{A}}$: there is a system $\langle \underline{\underline{B}}_i : i \in I \rangle$ of algebras such that $(\forall i \in I) \underline{\underline{B}}_i \in K$ and $\underline{\underline{A}} = P_{i \in I}\, \underline{\underline{B}}_i$ and the universe A of $\underline{\underline{A}}$ is nonempty$\}$.

Then $\mathbf{P}\, K \stackrel{df}{=} \mathbf{IP}^g\, K$.

The investigations in this paper can be carried over to the case when we consider algebras with empty universes, too.

Definition (Pigozzi [5]) . Let Q_1, Q_2 be two sequences of the letters $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}^r$, $\mathbf{P}^g$, $\mathbf{I}$. Then $Q_1 \leq Q_2$ is defined to hold iff for every class K of similar algebras we have $Q_1\, K \subseteq$ $\subseteq Q_2\, K$.

$Q_1 = Q_2$ is defined as ($Q_1 \leq Q_2$ and $Q_2 \leq Q_1$).

E.g. $\mathbf{SP} \geq \mathbf{PS}$ iff for every class K of similar algebras $\mathbf{SP}\, K \supseteq$ $\supseteq \mathbf{PS}\, K$.

The above definition is explained in more detail e.g. in the following parts of Grätzer [2]: § 23 on p. 154, Ex. 80 on p. 158, Problems 24, 28 on p. 161.

Remark: The note in [3] following Def. 0.3.1 gives reasons for using the operator $\mathbf{P}$ instead of $\mathbf{P}^g$. Theorem 1 and Theorem 2 below may be an additional reason.

Theorem 2 states that "$\mathbf{PS} \leq \mathbf{SP}$" holds without the Axiom of Choice. As a contrast, Theorem 1 says that the assumption "$\mathbf{P}^g\mathbf{S} \leq \mathbf{SP}^g$" implies the Axiom of Choice. ZF stands for Zermelo Fraenkel Set Theory and AC denotes the Axiom of Choice.

Theorem 1. Assume ZF. The statement "$\mathbf{SP}^g \geq \mathbf{P}^g\mathbf{S}$" is equivalent to AC. I.e. $ZF \cup \{``SP^g \geq P^gS"\} \vdash AC$ and

$$ZF \cup \{AC\} \vdash ``SP^g \geq P^gS".$$

Proof of Theorem 1: In the proof we shall apply the algebraic operators P^g, S to sets (without operations). This is done as follows.

We choose the similarity type t of our algebras to be empty and then the algebras of type t are sets without operations. More precisely, by Def. 0.1.5 of [3] a similarity type is a function $t: Op \longrightarrow \omega$ where Op is an arbitrary set. Then we choose $Op = 0$ and hence $t = 0$. Clearly 0 is a similarity type by the quoted definition. By Def. 0.1.1 of [3] the algebras of similarity type 0 are pairs $\langle A,0\rangle$ where A is an arbitrary nonempty set. Then we identify the pair $\langle A,0\rangle$ with the set A. We can do this because for any $B \subseteq A$ we have $\langle B,0\rangle \subseteq \langle A,0\rangle$ and for any $\langle A_i : i \in I\rangle$ we have $\langle P_{i\in I}A_i, 0\rangle = P_{i\in I}\langle A_i, 0\rangle$ where $P_{i\in I}A_i$ is the Cartesian product of the sets A_i, see [3] p. 29.

Throughout the proof, by a system $\langle X_i : i \in I\rangle$ we mean a set $\{\langle i, X_i\rangle : i \in I\}$ of pairs. I.e. a system is a function and this function is always a set and never a proper class.

First we prove $ZF \cup \{``SP^g \geq P^gS"\} \vdash AC$. Assume "$SP^g \geq P^gS$". We show that then AC holds.

Let $F = \langle X_i : i \in I\rangle$ be a family of nonempty sets. We have to show the existence of a "choice-function" for F, i.e. we have to show the existence of a function $f: I \longrightarrow \bigcup_{i\in I} X_i$ with the property that $(\forall i \in I) f(i) \in X_i$.

Let $K \overset{df}{=} \{x \cup \{\langle F,i\rangle\} : i \in I \text{ and } x \in X_i\}$. Then K is a set of similar algebras, namely every element of K is an algebra of type 0. For every $i \in I$ we have $\{\langle F,i\rangle\} \in S\,K$ since by $X_i \neq 0$ there is an $x \in X_i$ and then $\{\langle F.i\rangle\} \subseteq x \cup \{\langle F.i\rangle\} \in K$. Let

$A \stackrel{df}{=} P_{i\in I}\{\langle F,i\rangle\}$. Then $A \in \mathbf{P^g S}\, K$ since $A \neq 0$ by $A = \{\langle\langle F,i\rangle : i\in I\rangle\}$. Then $A \in \mathbf{SP^g}\, K$ by our assumption $\mathbf{SP^g} \geq \mathbf{P^g S}$. $A \in \mathbf{SP^g}\, K$ means that there are a set J and a system $\langle B_j : j\in J\rangle$ of elements of K such that $A \subseteq P_{j\in J} B_j$. By $A = \{\langle\langle F,i\rangle : i\in I\rangle\}$ we then have $\langle\langle F,i\rangle : i\in I\rangle \in P_{j\in J} B_j$ which means that J=I and $\langle F,i\rangle \in B_i$ for every $i\in I$.

Let $f \stackrel{df}{=} \langle B_i \sim \{\langle F,i\rangle\} : i\in I\rangle$. Then f is a set since $\langle B_i : i\in I\rangle$ is a set. f is also a function with domain I. Let $i\in I$. By the Axiom of Foundation (which is included in ZF) we have $(\forall x\in X_j)\ \langle F,i\rangle \notin x \cup \{\langle F,j\rangle\}$ for distinct $i,j\in I$. Therefore $\langle F,i\rangle \in B_i \in K$ implies $B_i = x \cup \{\langle F,i\rangle\}$ for some $x\in X_i$. Then $B_i \sim \{\langle F,i\rangle\} \in X_i$ since $(\forall x\in X_i)\langle F,i\rangle \notin x$ by the Axiom of Foundation. Hence f <u>is</u> a choice function for F, i.e. $f: I \to \bigcup_{i\in I} X_i$ such that $(\forall i\in I) f(i)\in X_i$.

We have proved that "$\mathbf{P^g S} \leq \mathbf{SP^g}$" implies AC. The other direction, $ZF\cup\{AC\} \vdash$ "$\mathbf{SP^g} \geq \mathbf{P^g S}$" is proved in Grätzer [2] as Thm. 23.1.

QED (Theorem 1)

Theorem 2. Assume ZF without AC. Then (i) and (ii) below hold.

(i) $\mathbf{SP} \geq \mathbf{PS}$

(ii) $\mathbf{SP^r} \geq \mathbf{P^r S}$.

Proof of Theorem 2: Notations: We shall use the notation of the monograph [3]. E.g. if X is a set then Sb X is the set of all subsets of X. Similarly $P_{i\in I}A_i$, $P_{i\in I}\underline{A}_i$, ${}^I K$, $J \upharpoonright f$, one-one function can be found in the "Index of symbols" (and in the "Index of Names and Subjects") at the end of [3]. We shall also use from [3] the following notation: Let $\langle \underline{A}_i : i\in I\rangle$ be a system of algebras in K, i.e. $\langle \underline{A}_i : i\in I\rangle \in {}^I K$. Then $\underline{\underline{A}} \stackrel{df}{=} \langle \underline{A}_i : i\in I\rangle$ and

$P\underline{\underline{A}} \stackrel{df}{=} P_{i\in I}\underline{\underline{A}}_i$. In short: If $\underline{\underline{A}} \in {}^I K$ then $\underline{\underline{A}} = \langle \underline{\underline{A}}_i : i \in I\rangle$ and $P\underline{\underline{A}} = P_{i\in I}\, \underline{\underline{A}}_i$. Let $\underline{\underline{A}} \in {}^I K$. Then $A \stackrel{df}{=} \langle A_i : i \in I\rangle$ is the system of the universes of the algebras in $\underline{\underline{A}}$, and $PA \stackrel{df}{=} P_{i\in I} A_i$. For more detailed explanation see the monograph [3].

<u>End of Notations</u>

<u>Proof of (ii)</u>: Let K be a class of similar algebras. Let $\underline{\underline{C}} \in \mathbf{P^r S}\, K$. We have to show $\underline{\underline{C}} \in \mathbf{SP^r}\, K$. $\underline{\underline{C}} \in \mathbf{P^r S}\, K$ means that there are a set I, a function $\underline{\underline{A}} \in {}^I(\mathbf{S}\, K)$ and a filter D on I such that $\underline{\underline{C}} \cong P\underline{\underline{A}}/D$. Let I, $\underline{\underline{A}}$ and D with the above properties be fixed. We define a class L and a relation $R \subseteq I \times L$ as follows:

$L = \{\langle i, \underline{\underline{B}}\rangle \in I \times K : \underline{\underline{A}}_i \subseteq \underline{\underline{B}}\}$ and

$R = \{\langle i, \langle i, \underline{\underline{B}}\rangle\rangle : \langle i, \underline{\underline{B}}\rangle \in L\}$.

Using the terminology of Levy [4], R is a relation with domain I, i.e. $(\forall i \in I)(\exists j \in L)\ \langle i, j\rangle \in R$. Then by II.7.11 of [4], there is a <u>set</u> $J \subseteq L$ such that $(\forall i \in I)(\exists j \in J)\ \langle i, j\rangle \in R$. (II.7.11 of [4] is true without AC since it is proved there to hold without AC.)

Denote the second projection on J by $\underline{\underline{G}}$, i.e.

$\underline{\underline{G}} \stackrel{df}{=} \langle \underline{\underline{B}} : \langle i, \underline{\underline{B}}\rangle \in J\rangle$.

Clearly, the function $\underline{\underline{G}}$ is a <u>set</u> (of pairs) since J is a set. Therefore $\underline{\underline{G}}$ is a system $\langle \underline{\underline{G}}_j : j \in J\rangle$ and $\underline{\underline{G}} \in {}^J K$. Note that $(\forall i \in I)(\exists \underline{\underline{B}})\, \langle i, \underline{\underline{B}}\rangle \in J$ and $(\forall \langle i, \underline{\underline{B}}\rangle \in J)\ \underline{\underline{A}}_i \subseteq \underline{\underline{G}}_{\langle i, \underline{\underline{B}}\rangle}$. Next we define a filter E on J.

$E \stackrel{df}{=} \{Y \in Sb\, J : (\exists X \in D)\ \{\langle i, \underline{\underline{B}}\rangle \in J : i \in X\} \subseteq Y\}$.

The above definition of E is an explicit definition of a subset of the powerset SB J of J.

E is a filter on J because D is a filter on I and $\{\langle i, \underline{\underline{B}}\rangle \in J : i \in X\} \cap \{\langle i, \underline{\underline{B}}\rangle \in J : i \in Z\} = \{\langle i, \underline{\underline{B}}\rangle \in J : i \in X \cap Z\}$, for every X, Z.

We shall show that $P\underline{\underline{A}}/D \cong | \subseteq PG/E$.

First we construct a homomorphism $g: P\underline{\underline{A}} \to P\underline{\underline{G}}$. Let $f \in PA$ be arbitrary. We define $g(f)$ as $g(f) \stackrel{df}{=} \langle f(i): \langle i,\underline{\underline{B}}\rangle \in J\rangle$. Clearly $g(f) \in PG$ since $g(f)(\langle i,\underline{\underline{B}}\rangle) = f(i) \in A_i \subseteq G_{\langle i,\underline{\underline{B}}\rangle}$ for every $\langle i,\underline{\underline{B}}\rangle \in J$. Hence g is a function from PA to PG. We show that g is a homomorphism:

Let m be an n-ary function symbol in the similarity type of the algebras in the class K. Let $\underline{\underline{M}}$ be an algebra similar to the elements of K. Then $m_{(\underline{\underline{M}})}$ is the interpretation of the operation symbol m in the algebra $\underline{\underline{M}}$. I.e. $m_{(\underline{\underline{M}})}$ is the operation of $\underline{\underline{M}}$ associated to the operation symbol m. Let $f_1,\ldots,f_n \in PA$.

$$g(m_{(P\underline{\underline{A}})}(f_1,\ldots,f_n)) = g(\langle m_{(\underline{\underline{A}}_i)}(f_1(i),\ldots,f_n(i)): i \in I\rangle) =$$
$$= \langle m_{(\underline{\underline{A}}_i)}(f_1(i),\ldots,f_n(i)): \langle i,\underline{\underline{B}}\rangle \in J\rangle = \langle m_{(\underline{\underline{G}}_j)}(g(f_1)j,\ldots,g(f_n)j): j \in J\rangle = m_{(P\underline{\underline{G}})}(g(f_1),\ldots,g(f_n)).$$

We have seen that g is a homomorphism $g: P\underline{\underline{A}} \to P\underline{\underline{G}}$.

Using this g now we construct a one-one homomorphism $h: P\underline{\underline{A}}/D \rightarrowtail P\underline{\underline{G}}/E$. We define h as $h \stackrel{df}{=} \{\langle f/D, g(f)/E\rangle : f \in PA\}$. Clearly $h \subseteq (PA/D) \times (PG/E)$ is a set of pairs. We show that h is a function:

Suppose $f/D = f_1/D$. We shall show that $g(f)/E = g(f_1)/E$. $f/D = f_1/D$ means that $(\exists X \in D)\ X \upharpoonright f \subseteq f_1$. Let $Y \stackrel{df}{=} \{\langle i,\underline{\underline{B}}\rangle \in J: i \in X\}$. Then $Y \in E$ and $Y \upharpoonright g(f) \subseteq g(f_1)$. This means $g(f)/E = g(f_1)/E$. We have seen that h is a function $h: PA/D \to PG/E$.

Next we show that h is a homomorphism. We shall use the fact that g is a homomorphism $g: P\underline{\underline{A}} \to P\underline{\underline{G}}$. Let m be an n-ary function symbol in the similarity type of the algebras in K. Let $f_1,\ldots,f_n \in PA$.

$h(m_{(P\underline{\underline{A}}/D)}(f_1/D,\dots,f_n/D)) = h(m_{(P\underline{\underline{A}})}(f_1,\dots,f_n)/D) =$

$= g(m_{(P\underline{\underline{A}})}(f_1,\dots,f_n))/E = m_{(P\underline{\underline{G}})}(g(f_1),\dots,g(f_n))/E =$

$= m_{(P\underline{\underline{G}}/E)}(g(f_1)/E,\dots,g(f_n)/E) = m_{(P\underline{\underline{G}}/E)}(h(f_1/D),\dots,h(f_n/D)).$

We show that the homomorphism h is one-one:

Suppose $h(f/D) = h(f_1/D)$ for some $f,f_1 \in PA$. This means $g(f)/E = g(f_1)/E$, i.e. $Y \upharpoonleft g(f) \subseteq g(f_1)$ for some $Y \in E$. By $Y \in E$ there is $X \in D$ such that $\{\langle i,\underline{\underline{B}}\rangle \in J : i \in X\} \subseteq Y$. Let $i \in X$. Then by the definition of J, $(\exists \underline{\underline{B}})\ \langle i,\underline{\underline{B}}\rangle \in J$. Then $\langle i,\underline{\underline{B}}\rangle \in Y$ by $i \in X$ and therefore $f(i) = g(f)(\langle i,\underline{\underline{B}}\rangle) = g(f_1)(\langle i,\underline{\underline{B}}\rangle) = f_1(i)$.

We have seen $X \upharpoonleft f \subseteq f_1$. Then $f/D = f_1/D$ by $X \in D$. Clearly this h is then an isomorphism of $P\underline{\underline{A}}/D$ into a subalgebra of $P\underline{\underline{G}}/E$ (without AC of course). By these we have seen that $P\underline{\underline{A}}/D \cong|\subseteq P\underline{\underline{G}}/E$. By $\underline{\underline{C}} \cong P\underline{\underline{A}}/D$ and $\underline{\underline{G}} \in {}^J K$ then $\underline{\underline{G}} \in \mathbf{IISP^r}\ K$.

Lemma 0. Assume ZF, without AC. Then each of the following statements is true (without AC).

$\mathbf{II} = \mathbf{I}$, $\mathbf{SS} = \mathbf{S}$, $\mathbf{IS} = \mathbf{SI}$, $\mathbf{IP^r} = \mathbf{P^r}$, $\mathbf{IP} = \mathbf{P}$.

Proof of Lemma 0: The proofs of the above statements are straightforward, even without AC. $\mathbf{IS} = \mathbf{SI}$ is proved as 0.2.15 of [3], and in the proof there it is emphasized that AC was not used.

QUED (Lemma 0)

By Lemma 0 above we have that $\mathbf{IISP^r} = \mathbf{ISP^r} = \mathbf{SIP^r} = \mathbf{SP^r}$. Therefore $\underline{\underline{C}} \in \mathbf{SP^r}\ K$ by $\underline{\underline{C}} \in \mathbf{IISP^r}\ K$.

Proof of (i): Let K be a class of similar algebras and let $\underline{\underline{C}} \in \mathbf{PS}\ K$. Then $\underline{\underline{C}} \cong P\underline{\underline{A}}$ for some system $\underline{\underline{A}} \in {}^I(\mathbf{S}\ K)$. Clearly $\{I\}$ is a filter on I and $\langle \{f\} : f \in PA\rangle$ is an isomorphism between $P\underline{\underline{A}}$ and $P\underline{\underline{A}}/\{I\}$.

In the proof of (ii), to I, $\underline{\underline{A}}$, and $D = \{I\}$ we constructed a set J, a system $\underline{\underline{G}} \in {}^{J}K$ and a filter E on J such that $P\underline{\underline{A}}/\{I\} \cong | \subseteq P\underline{\underline{G}}/E$. By the construction of E, if $D = \{I\}$ then $E = \{J\}$, and therefore $P\underline{\underline{G}}/E \cong P\underline{\underline{G}}$. We have $\underline{\underline{C}} \cong P\underline{\underline{A}} \cong P\underline{\underline{A}}/\{I\} \cong | \subseteq$ $\subseteq P\underline{\underline{G}}/\{J\} \cong P\underline{\underline{G}}$ and $\underline{\underline{G}} \in {}^{J}K$. Therefore $\underline{\underline{C}} \in \mathbf{IIISP}\,K$. By Lemma 0 then $\underline{\underline{C}} \in \mathbf{SP}\,K$.

<u>QUED</u> (Theorem 2)

Related results can be found in [1] and [5].

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