Commentationes Mathematicae Universitatis Carolinae

Bernhard Kawohl

On Liouville theorems, continuity and Hölder continuity of weak solutions to some quasilinear elliptic systems

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 4, 679--697

Persistent URL: `http://dml.cz/dmlcz/106034`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,4 (1980)

ON LIOUVILLE THEOREMS, CONTINUITY AND HÖLDER CONTINUITY OF WEAK SOLUTIONS TO SOME QUASILINEAR ELLIPTIC SYSTEMS

B. KAWOHL

(dedicated to Jindřich Nečas on the occasion of his 50th birthday)

Abstract: We prove that every bounded weak solution of a quasilinear elliptic system (0.1) is Hölder continuous in Ω if and only if the system has a Liouville-type property $\mathcal{L}(\mathbb{R}^n)$. The proofs are based on recent results of M. Giaquinta and J. Nečas.

Key words: Regularity, weak solution, quasilinear elliptic system, Liouville's property, blow up technique, Sobolev space.

Classification: 35J60

---

§ 0. Introduction. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. We consider the quasilinear elliptic system

$$(0.1) \quad \int_\Omega [a_{ij}^{rs}(x,u) \frac{\partial u_r}{\partial x_i} \frac{\partial \phi_s}{\partial x_j} + a_j^{rs}(x,u) \frac{\partial u_r}{\partial x_j} \phi_s] dx = \int_\Omega [g_j^r \frac{\partial \phi_r}{\partial x_j} + g^r \phi_r] dx, \qquad \phi \in [\mathcal{D}(\Omega)]^m,$$

where $r,s=1,2,\dots,m$; $i,j=1,2,\dots,n$; $u = (u_1,u_2,\dots,u_m) \in [W^{1,2}(\Omega) \cap L^\infty(\Omega)]^m$ and where the summation convention is used for r,s,i,j throughout the paper. The coefficients $a_{ij}^{rs}(x,u)$ and $a_j^{rs}(x,u)$ are continuous functions on $\Omega \times \mathbb{R}^m$, $g_j^r \in L^p(\Omega)$, $g^r \in L^{p/2}(\Omega)$, $p > n$, and the system (0.1) is strongly elliptic, i.e. there exists a $\mu > 0$ such that

$$(0.2) \quad a_{ij}^{rs}(x,u)\, \xi_i^r \xi_j^s \geq \mu |\xi|^2 \text{ holds for every } x \in \Omega,\ \xi \in \mathbb{R}^{mn}$$

and $u \in \mathbb{R}^m$.

As e.g. H. Frehse pointed out in [5], "the experience in the theory of elliptic equations suggests that regularity theorems hold for those types of equations, for which a Liouville-theorem is true". We are going to show that for systems of type (0.1),(0.2) regularity theorems are equivalent to Liouville-theorems. To be more precise, the properties $(\tilde{R})$ and $\tilde{\mathcal{L}}(\mathbb{R}^n)$ defined below are equivalent to each other. As a byproduct we prove that bounded continuous solutions of system (0.1),(02) are automatically Höldercontinuous.

$W^{k,p}(\Omega)$ denotes as usual the Sobolev space of those functions, whose derivatives of order up to k belong to the Lebesgue space $L^p(\Omega)$. $\mathcal{D}(\Omega)$ is the space of smooth testfunctions with compact support in Ω, and $C^{\alpha}(\Omega)$ consists of those continuous functions on Ω, which are locally α-Höldercontinuous. For convenience we shall write $W^{k,p}(\Omega)$, $L^p(\Omega)$, $\mathcal{D}(\Omega)$ and $C^{\alpha}(\Omega)$ from now on also for vectorvalued functions.

Let us point out that we assume the boundedness of weak solutions throughout the paper. According to [18] it is natural to start from L^{∞}-solutions. Nevertheless it should be interesting to investigate conditions under which the assumption $u \in L^{\infty}(\Omega)$ can be dropped. One step in that direction was recently done by E. Giusti and G. Modica in [15].

In order to present our main result in a concise form let us introduce a few more notations and definitions:
[M] denotes the family of those solutions $v \in W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ to system (0.1),(0,2), whose L^{∞}-norm is less than or equal to M,
[G] denotes the family of functions $g_j^r \in L^p(\Omega)$, $g^r \in L^{p/2}(\Omega)$

($j=1,2,\dots,n$; $r=1,2,\dots,m$; $p>n$) satisfying

$$\sum_{j,r} \|g_j^r\|_{L^p(\Omega)} + \sum_r \|g^r\|_{L^{p/2}(\Omega)} \le G,$$

[MG] is the union of those two sets, i.e. $[MG] = [M] \cup [G]$.
$A = A(M)$ is the following constant defined by the coefficients of the system (0.1)

$$A := \sup_{\substack{|\xi| \le M \\ x\in\Omega}} \Big[\sum_{i,j,r,s} |a_{js}^{rs}(x,\xi)| + \sum_{j,r,s} |a_j^{rs}(x,\xi)| \Big].$$

Definition of $\tilde{L}(\mathbb{R}^n)$: We say that the system (0.1) has Liouville's property $\tilde{L}(\mathbb{R}^n)$ if and only if for every $x^0 \in \Omega$ any solution $v \in W^{1,2}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ of the system

$$(0.3) \qquad \int_{\mathbb{R}^n} a_{ij}^{rs}(x^0,v) \frac{\partial v_r}{\partial x_i} \frac{\partial \phi_s}{\partial x_j}\, dx = 0, \qquad \phi \in \mathcal{D}(\mathbb{R}^n),$$

has to be a constant.

Definition of $(\tilde{C})$: We say that the system (0.1),(0.2) has property $(\tilde{C})$ if and only if every bounded weak solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (0.1) is locally continuous in Ω and if the modulus of continuity is uniform with respect to [M].

Definition of $(\tilde{R})$: We say that the system (0.1),(0.2) has property $(\tilde{R})$ if and only if every bounded weak solution $u \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (0.1) is Höldercontinuous in Ω with Hölder exponent $\alpha = \min\{\frac{1}{2}, 1 - \frac{n}{p}\}$, and if the following a priori estimate holds for every $\overline{\Omega'} \subset \Omega$:

$$\|u\|_{C^\alpha(\overline{\Omega'})} \le c(M,G,A,\mu,\Omega', \operatorname{dist}(\Omega', \partial\Omega)),$$

where the constant c is uniform with respect to [GM].

Main result: Under the general assumptions on Ω and the system (0.1),(0.2) (cf. the first paragraph of this introduction) the properties $\tilde{L}(\mathbb{R}^n)$, $(\tilde{C})$ and $(\tilde{R})$ are equivalent to each other

Let us make a few remarks about the history of the regularity-problem for quasilinear systems, which goes all the way back to Hilbert's 19th problem (see e.g. [22, 18, 6]). It was already known that $(\tilde{R})$ holds for n=2, i.e. for twodimensional domains (cf. [21, 22, 25]) or for m=1, i.e. for the case of a single equation (cf. [3, 18]). Höldercontinuity, however with undetermined Hölderexponent α, was also shown for systems with principle part in diagonal form (cf. e.g. [19], but also [27, 16, their case a=0]). As will be seen in § 2, Liouville's property holds in all these cases. Hence we obtain new proofs for already known theorems.

Nevertheles properties $(\tilde{C})$,$(\tilde{R})$ and $\tilde{L}(\mathbb{R}^n)$ do not always hold. In fact, based on an example by E. DeGiorgi [4], E. Giusti and M. Miranda [14] gave a counterexample of a quasilinear system with discontinuous solution $u(x) = \frac{x}{|x|}$ for $n \geq 3$. Observe that due to our main result the regularity-problem for solutions to system (0.1),(0.2) reduces to the equivalent, but simpler looking problem of verifying Liouville's property under suitable assumptions. In this context the important question arises: Which additional assumptions on the principal part of (0.1) are sufficient to provide $\tilde{L}(\mathbb{R}^n)$? As we mentioned above, we give a few positive answers in § 2.

There are many wellknown results concerning Hölder continuity almost everywhere in Ω . These are the so-called partial regularity results (cf. [23, 13, 10, 11, 8, 9, 6]). To be more explicit, let us recall the commonly used notation

$$u_{x^o,R} := \frac{1}{\operatorname{meas} B_R(x^o)} \int_{B_R(x^o)} u(x)dx$$

for the mean value of u in the ball $B_R(x^o)$ with center x^o and radius R and

$$U(x^o,R) := R^{-n} \int_{B_R(x^o)} |u(x) - u_{x^o,R}|^2 dx$$

as a measure for the mean deviation from $u_{x^o,R}$. The partial regularity result states essentially (cf. § 1) that a bounded weak solution u of system (0.1),(0.2) is Höldercontinuous in every point $x^o \in \Omega$ for which the following condition (0.4) holds:

(0.4) $\liminf_{R \to 0} U(x^o,R) = 0.$

Consequently, a bounded weak solution of system (0.1),(0.2) is Höldercontinuous everywhere in Ω, if (0.4) is satisfied for every $x^o \in \Omega$. If we denote the set of points $x \in \Omega$, in which (0.4) is violated, with S (standing for <u>singular</u> points) one is led to the question: When is S empty? To answer this question, additional a-priori-knowledge about the solution u of system (0.1),(0.2) seems to be needed. It is however desirable to have a replacement for (0.4) available that does not depend on the solution u of (0.1),(0.2). Property $\tilde{L}(\mathbb{R}^n)$ has precisely this advantage. Generally speaking we may say that Liouville's property enables us to close the gap between partial and global regularity.

The following diagram may illustrate how we proceed in the following paragraphs.

$$\begin{tikzcd}
(\tilde{C}) \arrow[d, Rightarrow, "\S\ 3"'] & (\tilde{R}) \arrow[l, Rightarrow, "\text{trivial}"'] \\
\tilde{L}(\mathbb{R}^n) \arrow[r, Rightarrow, "\S\ 2"'] & S = \emptyset \arrow[u, Rightarrow, "\S\ 1"']
\end{tikzcd}$$

As the reader will notice the main effort is hidden in § 1, especially in Lemma 1.5. We should also mention that $(\tilde{C})$ implies $S = \emptyset$ directly. The proof is left as an exercise to the reader.

Recent results [5,7,17,20,29,30,31] strongly suggest that results similar to the ones presented in this contribution ought to be expected for quasilinear systems "with quadratic growth", i.e. e.g. of type $D_\alpha(a_{\alpha\beta}(x,u,\nabla u)D_\beta u^i(x)) = f^i(x,u,\nabla u)$, with $f(x,u,p) \le a|p|^2 + b$.

Many of the ideas in this paper, especially the "hard part" § 1, are along the lines of M. Giaquinta and J. Nečas [8, 9, 26] who derived related results for the gradient of $W^{1,\infty}$-solutions to nonlinear systems. In fact the equivalence of $\tilde{L}(\mathbb{R}^n)$ and $(\tilde{R})$ was conjectured by J. Nečas. The author is deeply indebted to him for numerous stimulating discussions and advise.

This paper was written while the author enjoyed the hospitality of Charles University in Prague, where he participated in J. Nečas' seminar. Support was provided by Charles University, the Technical University of Darmstadt and the DAAD. Finally the author acknowledges H. Grabmüller's and M. Grüter's helpful comments on the manuscript.

§ 1. <u>A partial regularity result</u>. Results of this type were shown before, however under slightly stronger assumptions.

The following proposition is due to M. Giaquinta and J. Nečas, who used essentially the same method as E. Giusti et al. in their former papers. A proof of the local Hölder continuity can be found in [8],[9]. J. Nečas sketched the proof of the uniform a-priori-estimate in [26] and gave it more elaborately in his seminar-talks in October 1979.

Proposition 1.1. Let $u \in [M]$ be a bounded weak solution of system (0.1),(0.2). For every point $x^o \in \Omega$ such that

(1.1) $\liminf_{R \to o} U(x^o, R) = 0$

holds, there exists a ball $B_{R_1}(x^o) \subset\subset \Omega$ with $R_1 < \frac{1}{2} \operatorname{dist}(x^o, \partial\Omega)$ such that $u \in C^{\alpha}(\overline{B_{R_1}(x^o)})$ with $\alpha = \min\{\frac{1}{2}, 1 - \frac{n}{p}\}$ and such that the a-priori-estimate

(1.2) $\|u\|_{C^{\alpha}(\overline{B_{R_1}(x^o)})} \leq c(M, G, A, \mu, R_1)$

holds uniformly with respect to the class [MG].

Corollary 1.2. Let $u \in [M]$ be a bounded weak solution of system (0.1),(0.2). Suppose that for every $\overline{\Omega'} \subset \Omega$ property (1.1) holds uniformly with respect to $x^o \in \overline{\Omega'}$. Then $u \in C^{\alpha}(\Omega)$ with $\alpha = \min\{\frac{1}{2}, 1 - \frac{n}{p}\}$ and the a-priori-estimate

(1.3) $\|u\|_{C^{\alpha}(\Omega)} \leq c(M, G, A, \mu, \Omega', \operatorname{dist}(\Omega', \partial\Omega))$

holds uniformly with respect to the class [MG].

Remark: It can be shown that the set $S := \{x \in \Omega \mid \liminf_{R \to o} U(x, R) > 0\}$ of singular points is small in the following sense. Let $\mathcal{H}^{n-2}$ denote the (n-2)-dimensional Hausdorff-measure. S is a nullset with respect to this measure, i.e. $\mathcal{H}^{n-2}(S) = 0$. We refer to [9] or [12] for details.

Now Proposition 1.1 will be derived in a number of steps, following M. Giaquinta and J. Nečas [8, 9, 26].

Lemma 1.3: Let b_{ij}^{rs} $(r,s=1,2,\dots,m;\ i,j=1,2,\dots,n)$ be constant coefficients. Let $u \in W_{loc}^{1,2}(B_1(0)) \cap L^2(B_1(0))$ be a weak solution to the strongly elliptic system

$$\int_{B_1(0)} b_{ij}^{rs} \frac{\partial u_r}{\partial x_i} \frac{\partial \phi_s}{\partial x_j}\, dx = 0, \qquad \phi \in \mathcal{D}(B_1(0)),$$

with the ellipticity constant $\nu > 0$, i.e. $b_{ij}^{rs}\eta_i^s\eta_j^s \geq \nu|\eta|^2$ for every $\eta \in \mathbb{R}^{mn}$. Then the inequality

(1.4) $\qquad U(0,\rho) \leq K\,\rho^2\, U(0,1)$

holds for $\rho \in (0,1)$, where K is a constant depending only on ν and $\max b_{ij}^{rs}$.

The original proof of this Lemma is apparently due to S. Campanato [2], a shorter proof was given in [8, 9].

Before we proceed to the next Lemmata we have to introduce a decomposition (1.8) of the solution u to the quasilinear system (0.1),(0.2), If $u \in [M]$ is such a solution, there exists an $\tilde{R} > 0$ depending on A and M (but otherwise not on u) such that for every $x^o \in \Omega$ and for every $R \leq R_2$, with

(1.5) $\qquad R_2 := \min\{\tilde{R}(A,M), \mathrm{dist}(x^o,\partial\Omega)\} = R_2(A,M,\mathrm{dist}(x^o,\partial\Omega)),$

the following linear elliptic system (1.6) (with L^∞-coefficients) for the unknown function w^R is uniquely solvable in $W_o^{1,2}(B_R(x^o))$:

$$(1.6)\qquad \int_{B_R(x^o)} [\, a_{ij}^{rs}(x,u) \frac{\partial w_r^R}{\partial x_i} \frac{\partial \phi_s}{\partial x_j} + a_i^{rs}(x,u) \frac{\partial w_r^R}{\partial x_i} \phi_s \,]\, dx$$

$$= \int_{B_R(x^o)} [\, g_j^r \frac{\partial \phi_r}{\partial x_j} + g^r \phi_r \,]\, dx, \qquad \phi \in \mathcal{D}(B_R(x^o)).$$

In fact, due to the ellipticity of the system and due to Friedrich's inequality

(1.7) $\int_{B_R(x^o)} |w^R|^2 dx \leq c\ R^2 \int_{B_R(x^o)} |\nabla w^R|^2 dx$ for $w^R \in$
$\in W^{1,2}(B_R(x^o))$

is the bilinear form $a(w^R, \phi)$ defined by the left hand side of system (1.6) is strictly coercive on $\mathring{W}^{1,2}(B_R(x^o))$ as long as $\tilde{R}(A,M)$ is sufficiently small. Let us stress the point that R_2 (defined by (1.5)) is independent of the class [GM] and depends on x^o only via dist $(x^o, \partial\Omega)$. Since (1.6) is uniquely solvable for $R \leq R_2$ we may decompose any solution u of the quasilinear system (0.1),(0.2) in the following manner:

(1.8) $u = v^R + w^R$, where $w^R \in \mathring{W}^{1,2}(B_R(x^o))$ solves system (1.6).

Now we investigate v^R and w^R as $R \to 0$.

Lemma 1.4: Let w^R be defined as above with $R < R_2$. There exists a constant c depending on G, A, μ and R_2 such that the following holds uniformly with respect to $x^o \in \Omega' \subset\subset \Omega$ and uniformly with respect to the class [GM]:

(1.9) $W^R(x^o, R) \leq c(G, A, \mu, R_2)\ R^{2-2n/p}$.

Proof: Since $\overline{\mathcal{D}(B_R(x^o))} = \mathring{W}^{1,2}(B_R(x^o))$ we may set $\phi = w^R$ in (1.6). The strict coerciveness of $a(w^R, w^R)$, inequality (1.7) and Hölder's inequality imply

$$\|\nabla w^R\|^2_{L^2(B_R(x^o))} \leq c(G, A, \mu, R_2)\ R^{n/2-n/p}$$

and hence (1.9) holds.

Unfortunately the function v^R from the decomposition (1.8) has not the same "nice" behaviour. Nevertheless we have at least the following result (cf. (1.4)).

Lemma 1.5. Let $u \in [M]$ be a bounded weak solution of system (0.1),(0.2) and let $\tau \in (0,1)$ be fixed. Then there exists $\varepsilon_o = \varepsilon_o(\tau, M) > 0$ and $R_o = R_o(\tau, M)$ such that for every $x^o \in \Omega$ and for every $R \leq \{R_o(\tau, M), \operatorname{dist}(x^o, \partial\Omega)\}$ the implication

$$V^R(x^o, R) < \varepsilon_o^2 \Rightarrow V^R(x^o, \tau R) \leq 2K\, \tau^2\, V^R(x^o, R)$$

holds, where $K = K(M, A)$ is the constant form (1.4) for the family of coefficients $a_{ij}^{rs}(x^o, \xi)$, $x^o \in \Omega$, $|\xi| \leq M$.

This Lemma is a modification of the analogous "main Lemmas" in [23,13,10,12]. A detailed proof was given in [8, 9].

Now we are able to prove Proposition 1.1. Recalling (1.5) we choose $R_1 < \frac{1}{2} R_2$. In order to derive (1.2) it suffices to show (cf.[1])

(1.10) $\quad U(\bar{x}, \rho) \leq c\rho^{2\alpha}$ for every $\bar{x} \in \overline{B_{R_1}(x_o)}$ and for every $\rho > 0$

in some sufficiently small neighborhood of zero, where the constant $c(R_1, M, G, A, \mu)$ is independent of $\bar{x}$ and ρ.

So choose $\bar{x} \in \overline{B_{R_1}(x^o)}$ and $R_{\bar{x}} < R_1$. Then $\overline{B_{R_x}(\bar{x})} \subset B_{R_2}(x^o)$. Recalling (1.8) yields

$$V^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) \leq 2\, U(\bar{x}, R_{\bar{x}}) + 2\, W^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) \leq 2\, U(x^o, 2R_1) + 2\, W^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}).$$

By assumption (1.1) $U(x^o, 2R_1) \longrightarrow 0$ as $R_1 \longrightarrow 0$, and the last term converges to zero uniformly with respect to $\bar{x} \in \overline{B_{R_1}(x^o)}$. This follows from Lemma 1.4. Hence after a possible change of R_1 (which may depend on x^o unless (1.1) holds uniformly)

(1.11) for every $\varepsilon_o > 0$ there exists a sufficiently small $R_{\bar{x}}(\varepsilon_o) \in (0, R_1)$ (but $R_{\bar{x}}$ independent of $\bar{x}$) such that $V^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) < \varepsilon_o^2$.

For the remainder of this proof choose $\tau \in (0, \frac{1}{2})$ such that $4K\tau < \frac{1}{2}$, with K from Lemma 1.5. According to this Lemma determine $\varepsilon_0(\tau, M)$ and due to (1.11) also $R_{\bar{x}}$ such that $V^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) < \varepsilon_0^2$. We want to find an estimate for $V^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}})$ in terms of τ. To this end observe that

$$(1.12) \qquad V^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) \le 2\{V^{R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) + W^{R_{\bar{x}}}(x, \tau R_{\bar{x}}) + \\ + W^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}})\}.$$

Due to Lemmata 1.4 and 1.5 the first and the last term on the right hand side can be extimated by τ^2 and $\tau^{2-2n/p}$ respectively, whereas a calculation similar to the proof of Lemma 1.4 yields

$$W^{R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) \le c(G, A, \mu, R_1)\, \tau^{2-n}\, R_{\bar{x}}^{2} - 2n/p,$$

so that (1.12) implies

$$(1.13) \qquad V^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) \le 4K\tau^2\, V^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) + 4c(G, A, \mu, R_1) R_{\bar{x}}^{2-2n/p}.$$

In order to reiterate (1.13) we have to show that also

$$(1.14) \qquad V^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) < \varepsilon_0^2.$$

However the choice of τ and (1.13) imply

$$V^{\tau R_{\bar{x}}}(\bar{x}, \tau R_{\bar{x}}) \le \varepsilon_0^2/4 + c\, R_{\bar{x}}^{2-2n/p},$$

hence after a possible change of R_1 the second term involving $R_{\bar{x}} \in (0, R_1)$ is sufficiently small. Recall that ε_0 was chosen independently of $R_{\bar{x}}$, so (1.14) does not hold and we may reiterate (1.13). By induction it can be shown that for every positive integer k we have

$$V^{\tau^k R_{\bar{x}}}(\bar{x}, \tau^k R_{\bar{x}}) \le (4K\tau^2)^k V^{R_{\bar{x}}}(\bar{x}, R_{\bar{x}}) + 4 R_{\bar{x}}^{2-2n/p} c(\tau) (\tau^k + \tau^{k(2-2n/p)})$$

and

$$V^{\tau^k R_{\bar{x}}}(\bar{x}, \tau^k R_{\bar{x}}) \le c(\tau, \varepsilon_0, K, G, A, \mu, R_1) (\tau^k + \tau^{k(2-2n/p)})$$

holds with c independent of $\bar{x}$ and k. This and Lemma 1.4 yield

$$U(\bar{x}, \tau^k R_{\bar{x}}) \le 2 \{ V^{\tau^k R_{\bar{x}}}(\bar{x}, \tau^k R_{\bar{x}}) + W^{\tau^k R_{\bar{x}}}(\bar{x}, \tau^k R_{\bar{x}}) \}$$

$$\le c \{ \tau^k + \tau^{k(2-2n/p)} \}$$

with c independent of $\bar{x}$ and k, and consequently

(1.15) $\quad U(\bar{x}, \tau^k R_{\bar{x}}) \le c\, \tau^{2\alpha k}$, where $\alpha = \min\{\frac{1}{2}, 1 - \frac{n}{p}\}$.

Recall that we want to prove (1.12). Now choose ρ arbitrarily small and $k \in \mathbb{N}$ such that $\tau^{k+1} R_{\bar{x}} < \rho < \tau^k R_{\bar{x}}$. Clearly $\tau^k < \rho / \tau R_{\bar{x}}$, and so (1.1) implies the desired inequality (1.10).

§ 2. Liouville's property implies regularity $(\tilde{R})$, and some sufficient conditions for $\tilde{L}(\mathbb{R}^n)$

Theorem 2.1. Let $U \in [M]$ be a bounded weak solution of system (0.1),(0.2). If the system (0.1) has property $\tilde{L}(\mathbb{R}^n)$, then $(\tilde{R})$ holds. i.e. u is Höldercontinuous in Ω with Hölder-exponent $\alpha = \min\{\frac{1}{2}, 1 - \frac{n}{p}\}$, and for every $\overline{\Omega'} \subset \Omega$ the a priori estimate (1.3) holds.

Proof: According to Corollary 1.2 we only need to show that (1.1) holds uniformly with respect to $x^0 \in \overline{\Omega'}$. Let us assume in the contrary that there exists a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \overline{\Omega'}$, $x_k \to x_0$, and $\{R_k\} \subset \mathbb{R}^+$, $R_k \to 0$ such that

$U(x_k, R_k) > \varepsilon$. Now consider the following family of transformations $\{T_{R_k}\}_{k \in \mathbb{N}}$:

$$T_{R_k}: \begin{cases} (x,u) & \longrightarrow (y = \dfrac{x-x_k}{R_k},\ u_{R_k}(y) = u(x_k + R_k y)) \\ \Omega \times W^{1/2}(\Omega) \cap L^{\infty}(\Omega) & \longrightarrow \Omega_{R_k} \times W^{1/2}(\Omega_{R_k}) \cap L^{\infty}(\Omega_{R_k}) \end{cases}$$

<u>Remark 2.2</u>: Observe that Ω_R blows up as R tends to zero and that Ω_R shrinks as R tends to infinity. Using an analogy from optical lenses, taking R small amounts to looking at the neighbourhood of x^0 with a magnifying glass, and the whole family $\{T_R\}_{R>0}$ acts on Ω like a zoom-lense. If one observes an optical object through a zoom-lense at different focusses, its size appears different, but not its colour. The family $\{T_R\}_{R>0}$ has a similar property: The "size" of Ω_R and ∇u_R varies with R, but the L^{∞}-norm of u_R stays invariant. In fact $\|u\|_{L^{\infty}(\Omega)} = \|u_R\|_{L^{\infty}(\Omega_R)}$ for every $R > 0$. If one wants another norm of u to stay invariant under a similar family of transformations one has to modify the definition of T_R. This was done in [8, 9, 26] where $\|\nabla u\|_{L^{\infty}}$ stayed invariant. A technique of this type seems to be known as <u>blow-up-technique</u>.

We continue the proof of Theorem 2.1. Using the transformation T_{R_k} the system (0.1) can be rewritten as

$$(2.1) \quad \int_{\Omega_{R_k}} a_{ij}^{rs}(x_k + R_k y, u_{R_k}) \frac{\partial u^r_{R_k}}{\partial y_i} \frac{\partial \phi^s}{\partial y_j}\, dy$$

$$+ \int_{\Omega_{R_k}} a_j^{rs}(x_k + R_k y, u_{R_k}) \frac{\partial u^r_{R_k}}{\partial y_j} \phi_s R_k dy +$$

$$+ \int_{\Omega_{R_k}} g_j^r \frac{\partial \phi_r}{\partial y_j} R_k dy + \int_{\Omega_{R_k}} g^r \phi_r R_k^2\, dy$$

From the ellipticity of the system we obtain in a standard way for $B_{2a}(o) \subset \Omega_{R_k}$

$$\int_{B_a(o)} |\nabla_y u_{R_k}|^2 \, dy \leq c(a, A, \mu, G).$$

So $u_{R_k} \longrightarrow p$ weakly in $W^{1,2}(B_a(o))$. Hence for every $a > 0$ there exists a subsequence $\{u_{R_k}\}_{R_k \to 0}$ such that $u_{R_k} \longrightarrow p$ in $L^2(B_a(0))$ as well as point-wise a.e. on $B_a(0)$. Since T_{R_k} leaves the L^∞-norm of u invariant, $p \in L^\infty(\mathbb{R}^n)$ with $|p| \leq M$. If we send R_k to zero in (2.1), p solves the system

$$\int_{\mathbb{R}^n} [a_{ij}^{rs}(x^o, p) \frac{\partial p^r}{\partial y_i} \frac{\partial \phi^s}{\partial y_j}] \, dy = 0, \qquad \phi \in \mathcal{D}(\mathbb{R}^n).$$

This limiting procedure can be justified using the continuity of the coefficients and Lebesgue's dominated convergence theorem. Now Liouville's property implies that p = const, and using the fact that $u_{R_k} \longrightarrow p$ strongly in $L^2(B_a(0))$, we obtain

$$U(x_k, R_k) \leq R_k^{-n} \int_{B_{R_k}(x_k)} |u(x) - p|^2 \, dx = \int_{B_1(o)} |u_{R_k}(y) - p|^2 \, dy \to$$
$$\longrightarrow 0 \quad \text{(q.e.d.)}.$$

<u>Theorem 2.2</u>. Liouville's property $\tilde{L}(\mathbb{R}^n)$ holds for $n=2$, i.e. for plane domains.

<u>Proof</u>: Let $v \in W^{1,2}_{loc}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be a weak solution of the system (0.3). We have to show that v is constant. Let $T > 0$ be fixed and let $\eta \in \mathcal{D}(B_{2T}(o))$, where $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_T(o)$ and $|\nabla \eta| \leq \frac{C}{T}$. Furthermore put $\phi_s = \eta^2 v_s$. Using the ellipticity, equation (0.3) and Hölder's inequality we obtain

$$\mu \|\eta \nabla v\|^2_{L^2(B_{2T}(o))} \leq A \cdot \|v\|_{L^\infty} \cdot \frac{c}{T} \|\eta \nabla v\|_{L^2(B_{2T}(o))}$$
$$\cdot \{\text{meas } B_{2T}(o)\}^{1/2} \leq c \, \|\eta \nabla v\|_{L^2(B_{2T}(o))},$$

with c independent of T. This estimate works for n=2 only. $T \to \infty$ yields $\nabla v \in L^2(\mathbb{R}^2)$. But there exists a sequence $\psi^n \in \mathfrak{D}(\mathbb{R}^2)$ with $\nabla\psi^n \longrightarrow \nabla v$ in $[L^2(\mathbb{R}^2)]^{2m}$. Therefore

$\int_{\mathbb{R}^2} a_{ij}^{rs}(x^o,v) \frac{\partial v_r}{\partial x_i}\frac{\partial v_s}{\partial x_j}\, dx = 0$ holds, and hence v is constant.

<u>Theorem 2.3</u>. Liouville's property $\tilde{L}(\mathbb{R}^n)$ holds for m=1, i.e. for quasilinear equations.

J. Moser proved this theorem in [24, p. 465] for <u>linear</u> equations with L^∞-coefficients. His proof extends without any changes to our quasilinear equation.

<u>Theorem 2.4</u>. Liouville's property $\tilde{L}(\mathbb{R}^n)$ holds for systems with principal part in diagonal form. +)

<u>Proof</u>: System (0.3) takes the (diagonal) form

$$(2.2) \qquad \int_{\mathbb{R}^n} A_{ij}^{r}(x^o,v) \frac{\partial v_r}{\partial x_i}\frac{\partial \phi_r}{\partial x_j}\, dx = 0, \qquad \phi \in \mathfrak{D}(\mathbb{R}^n),$$

with $A_{ij}^r = \delta_{rs}\, a_{ij}^{rs}$. But system (2.2) is a system of m single equations for each component v_r of v. Hence Theorem 2.3 applies.

§ 3. <u>Continuity $(\tilde{C})$ implies Liouville's property</u>. In contrast to [26] we do not make use of a maximum principle for proving that already $(\tilde{C})$ and not only $(\tilde{R})$ implies $\tilde{L}(\mathbb{R}^n)$.

<u>Theorem 3.1</u>. Suppose that system (0.3) has property $(\tilde{C})$, i.e. every bounded weak solution u of system (0.3) is continuous in $\bar{\Omega}$ and the modulus of continuity of u depends on M but

----

+) For nondiagonal systems see [28, 29, 30].

otherwise not on u. Then Liouville's property $\tilde{L}(\mathbb{R}^n)$ holds.

Proof: Let $x^o \in \Omega$ be fixed and $v \in W^{1,2}_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of system (0.3). We intend to show that $v(x) \equiv \equiv v(x^o)$ for any $x \in \mathbb{R}^n$. To this end we choose a number a such that $B_a(x^o) \subset \Omega$ and $x \in \mathbb{R}^n$. Then there exists a sufficiently large R_o such that $y := \frac{x-x_o}{R} \in B_a(o)$ for every $R > R_o$. We define $v_R(y) = v(x^o + Ry)$. For every $R>0$ the function v_R is again a solution of (0.3). By assumption, its solutions are continuous. Hence for $y \in B_a(o)$ we have

(3.1) $\quad |v_R(y) - v_R(o)| \to 0$ as $y \to 0$ uniformly w.r.t. $R > R_o$.

Observe that it is important to assume a uniform modulus of continuity in $(\tilde{C})$. For fixed $x \in \Omega$ and for R tending to infinity, y converges to zero. Writing (3.1) in the x-ccordinates we obtain

$$|v(x) - v(x^o)| = 0, \text{ which completes the proof.}$$

Remark 3.2: In the proof of Theorem 3.1 we use again the blow-up-technique from § 2 [cf. Remark 2.2], however with $R \to \infty$. Here we applied T_R to the solution v of system (0.3) and used it to let $\mathbb{R}^n$ shrink inside Ω, whereas in § 2 we applied it to the solution u of system (0.1),(0.2) in order to spread Ω out over the whole $\mathbb{R}^n$. Essentially the same technique was used (in a different setting) by J. Frehse with his "friend" in [5].

References

[1] CAMPANARO S.: Proprietà di Hölderianità di alcune classe di funzioni, Ann. Sc. Norm. Sup. Pisa (3)17(1963), 175-188.

[2] CAMPANATO S.: Equazioni ellittiche del II. ordine e spazi $\mathcal{L}^{2,\lambda}$, Ann. Mat. Pura Appl. (4)69(1965), 321-382.

[3] De GIORGI E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3)3(1957), 25-43.

[4] De GIORGI E.: Un esempio di estremali discontinui per un problema variazionale di tipo ellitico, Boll. Un. Mat. Ital. (4)1(1968), 135-141.

[5] FREHSE J.: On two-dimensional quasi-linear elliptic systems, Manuscr. math. 28(1979), 21-49.

[6] GIAQUINTA M.: Sistemi ellitici non lineari. Teoria della regolarità, Boll. Un. Mat. Ital. (5)16A(1978), 259-283.

[7] GIAQUINTA M. and E. GIUSTI: Nonlinear elliptic systems with quadratic growth, Manuscr. math. 24(1978), 323-349.

[8] GIAQUINTA M. and J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems via Liouville's type property, Comment. Math. Univ. Carolinae 20(1979), 111-121.

[9] GIAQUINTA M. and J. NEČAS: On the regularity of weak solutions to nonlinear elliptic systems of partial differential equations, J. Reihe Angew. Math. 316 (1980), 140-159.

[10] GIUSTI E.: Regolarità parziale delle soluzioni di sistemi ellitici quasi lineari di ordine arbitraro, Ann. Sc. Norm. Sup. Pisa (3)23(1969), 115-141.

[11] GIUSTI E.: Un aggiunto alla mia nota: regolarità parziale delle soluzioni di sistemi ellitici quasi lineari di ordine arbitrario, Ann. Sci. Norm. Sup. Pisa (4)27(1973), 161-166.

[12] GIUSTI E.: Precisazione delle funzioni di $H^{1,p}$ e singolarità delle soluzioni deboli di sistemi ellitici non lineari, Boll. Un. Mat. Ital.(4)2(1969),71-76.

[13] GIUSTI E. and M. MIRANDA: Sulla regolarità delle soluzioni deboli di una classe di sistemi ellitici quasi-lineari, Arch. Mech. Anal. 31(1968), 173-184.

[14] GIUSTI E. and M. MIRANDA: Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, Boll. Un. Mat. Ital.(4)1(1968), 219-226.

[15] GIUSTI E. and G. MODICA: A note on regular points for solutions of elliptic systems, Manuscr. math. 29(1979), 417-426.

[16] HILDEBRANDT S. and K.O. WIDMAN: On the Hölder-continuity of weak solutions of quasilinear elliptic systems of second order, Ann. Sc. Norm. Sup. Pisa (4)4(1977), 145-178.

[17] HILDEBRANDT S. and K.O. WIDMAN: Sätze vom Liouvilleschen Typ für quasilineare elliptische Gleichungen und Systeme, Preprint 225, SFB 72, Bonn 1978.

[18] LADYZENSKAYA O.A. and N.N. URAL'TSEVA: Linear and quasilinear elliptic equations, Academic Press, New York 1968.

[19] LAIR V.A.: Quasilinear elliptic systems, Ann. Mat. Pura Appl.(4)116(1978), 17-56.

[20] MEIER M.: Liouville theorems for nonlinear elliptic equations and systems, Manuscr. math. 29(1979), 207-228.

[21] MOOREY C.B.: On the solutions of quasilinear elliptic differential equations, Trans. Amer. Math. Soc. 43 (1938), 126-166.

[22] MORREY C.B.: Existence and differentiability theorems for the solutions of variational problems for multiple integrals, Bull. Amer. Math. Soc. 46(1940), 439-458.

[23] MORREY C.B.: Partial regularity results for nonlinear elliptic systems, J. Math. Mech. 17(1968), 649-670.

[24] MOSER J.: A new proof of DeGiorgi's theorem concerning the regularity problem for elliptic differential

equations, Comm. Pure Appl. Math. 13(1960), 457-468.

[25] NEČAS J.: Sur la regularité des solutions variationelles des équations elliptiques non-linéaires d'ordre 2k en deux dimensions, Ann. Sc. Norm. Sup. Pisa (3)21(1967), 427-457.

[26] NEČAS J.: A necessary and sufficient condition for the regularity of weak solutions to nonlinear elliptic systems of partial differential equations, to appear in Abh. Akad. Wiss. DDR, Abt. Math. Naturw. Technik.

[27] WIEGNER M.: A priori Schranken für Lösungen gewisser elliptischer Systeme, Manuscr. math. 18(1976), 279-297.

[28] FREHSE J.: Essential selfadjointness of singular elliptic operators, Boll. Soc. Bras. Mat. 8.2(1977), 87-107.

[29] IVERT P.A.: On quasilinear elliptic systems of diagonal form, Math. Z. 170(1980), 283-286.

[30] MEIER M.: Liouville theorem for nondiagonal elliptic systems in arbitrary dimensions, to appear in Math.Z.

[31] WIEGNER M.: Regularity theorems for nondiagonal elliptic systems, Preprint no 338, SFB 72, Bonn (1980).

FB Mathematik der Technischen Hochschule

Schlossgartenstr. 7

D-6100 Darmstadt

Fed. Rep. of Germany

(Oblatum 10.12. 1979)