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ZEROS OF ACCRETIVE OPERATORS

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Abstract: We show how properties of the resolvent can be used to provide simple proofs of new results on existence of zeros and surjectivity for accretive operators.

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Classification: 47H06, 47H15

Let E be a real Banach space, and define the duality mapping J from E into the family of weak star compact convex subsets of E^* by

$$J(x) = \{x^* \in E^*: (x,x^*) = |x|^2 \text{ and } |x^*| = |x|\}.$$

Let $(y,x)_+ = \max\{(y,j): j \in J(x)\}$. Recall that a subset A of $E \times E$ with domain $D(A)$ and range $R(A)$ is said to be accretive if $(y_1 - y_2, x_1 - x_2)_+ \geq 0$ for all $[x_i, y_i] \in A$, $i = 1,2$. It is called m-accretive if, in addition, $R(I + rA) = E$ for some (hence all) $r > 0$. The resolvent J_r and the Yosida approximation A_r of A are defined by $J_r = (I + rA)^{-1}$ and $A_r = (I - J_r)/r$ respectively.

The purpose of this note is to show how properties of the resolvent can be used to provide simple proofs of new results on existence of zeros and surjectivity for accretive operators. The operators may be set-valued and no continuity assumptions are imposed on them. Theorems 3 and 4 provide necessary and sufficient conditions for the existence of zeros, and Theorem 6 is a general surjectivity result. Although our results are stated for m-accretive operators, this assumption can often be relaxed. For details concerning the fixed point property for nonexpansive mappings, see [7].

We begin with a lemma (cf. [9, Lemma 1.1]). Let $\|D\| = \inf\{|x| : x \in D\}$.

Lemma 1. Let E be a Banach space each bounded closed convex subset of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. If $y_n \in Ax_n$, $\{x_n\}$ is bounded, and $y_n \to y$, then $y \in R(A)$.

Proof. We may assume that $y = 0$. Let $R = \limsup_{n \to \infty} |x_n|$. The set $\{z \in E : \limsup_{n \to \infty} |z - x_n| \le R\}$ is non-empty, bounded, closed, and convex. Since $|J_r x_n - x_n| \le r \|Ax_n\| \le r|y_n| \to 0$, it is also invariant under J_r. Consequently, it contains a fixed point of J_r, hence a zero of A.

We use this lemma to provide a proof of the following result (cf. [10, Theorem 2] and [6, Theorem 1]).

Theorem 2. Let E be a Banach space each bounded closed convex subset of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. Then A is zero free if and only if $\lim_{t \to \infty} |J_t x| = \infty$ for each x in E.

Proof. If $y \in A^{-1}0$, then $|A_t x| \le 2|x - y|/t$, so that $|J_t x|$ is bounded. Conversely, if $\{x_n = J_{t_n} x\}$ is bounded for some x in E and some sequence $t_n \to \infty$, then $y_n = (x - x_n)/t_n \xrightarrow[n \to \infty]{} 0$. Since $y_n \in Ax_n$m the result follows from Lemma 1.

Theorem 2 will be used in the proof of our next result. Let cl(D) and bdy(D) denote the closure and boundary of a subset D of E.

Theorem 3. Let E be a Banach space each bounded closed convex subset of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. Then $0 \in R(A)$ if and only if there is a bounded open subset U of E and a point x_0 in $U \cap \mathrm{cl}\,(D(A))$ such that $(y, x - x_0)_+ \geq 0$ for all $x \in \mathrm{bdy}(U) \cap D(A)$ and $y \in Ax$.

Proof. If $0 \notin R(A)$, then $\lim_{t \to \infty} |J_t x_0| = \infty$ by Theorem 2. Since $\lim_{t \to 0+} J_t x_0 = x_0$ and $J_t x_0$ is a continuous function of t, there is a positive r such that $J_r x_0 \in \mathrm{bdy}(U)$. Therefore $(A_r x_0, J_r x_0 - x_0)_+ \geq 0$ and $x_0 = J_r x_0$, a contradiction. Necessity is obvious.

If, in addition, E is uniformly smooth, then by [11, Theorem 1] the strong $\lim_{t \to \infty} J_t x_0$ exists and belongs to $A^{-1}0$. Thus in this case we can conclude that A has a zero in cl(U).

Note that in contrast with Theorem 3 the sufficient condition of [6, Theorem 2] is certainly not necessary.

We now present two variants of Theorem 3 (both with a weaker assumption on E). Theorem 4 improves upon [8, Lemma 1.2].

Theorem 4. Let E be a Banach space the unit ball of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. Then $0 \in R(A)$ if and only if there is a positive $R > 0$ and a point x_0 in cl(D(A)) such that $(y, x - x_0)_+ \geq 0$ for all $y \in Ax$ with $|x - x_0| = R$.

Proof. Let $B = B(x_0, R) = \{x \in E : |x - x_0| < R\}$. Let $x \in B$.

Since $|J_r x - x_0| \leq |x-x_0| + |J_r x_0 - x_0|$, we see that $J_r x \in B$ for all sufficiently small positive r. If $|J_t x - x_0| = R$ for some t, then $(A_t x, J_t x - x_0)_+ \geq 0$. Therefore $R^2 = |x_0 - J_t x|^2 \leq$ $\leq (x-x_0, J_t x - x_0)_+ \leq |x-x_0| R$, and $|x-x_0| \geq R$, a contradiction. Thus we see that $J_t x \in B$ for all $0 < t < \infty$. It follows that for each fixed $r > 0$, J_r maps cl(B) into itself. Hence J_r has a fixed point, which is a zero of A.

In order to see that Theorem 4 is not true in all Banach spaces, define $T: c_0 \longrightarrow c_0$ by $T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots)$ and let $A = I - T$.

Theorem 5. Let E be a Banach space the unit ball of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. Assume that there are a bounded open subset U of E, a point x_0 in $U \cap \mathrm{cl}(D(A))$, and a positive c such that $(y, x-x_0)_+ \geq c$ for all $x \in \mathrm{bdy}(U) \cap D(A)$ and $y \in Ax$, and let $R = \sup\{|x-x_0| : x \in \mathrm{bdy}(U)\}$. Then $B(0, c/R) \subset R(A)$.

Proof. We first show that $0 \in R(A)$, and then apply this result to $A' \subset E \times E$ defined by $A'x = Ax - z$ with $|z| < c/R$. Let $U \subset B(x_0, R)$, and let r and z satisfy $r > R^2/c$ and $|z - x_0| \leq R$. Defining $C \subset E \times E$ by $Cx = Ax + (x_0 - z)/r$, we see that $(Cx, x-x_0)_+ \geq c - R^2/r > 0$ for $x \in \mathrm{bdy}(U)$. The proof of Theorem 3 shows that $J_t^C x_0$ remains in U for all $t > 0$. Since $J_r^C x_0 = J_r^A z$, we see that J_r^A maps $\mathrm{cl}(B(x_0, R))$ into itself. It follows that 0 is indeed in R(A). Now let A' be defined as above. Then A' satisfies the hypotheses of the theorem with $c' = c - |z| R > 0$. Therefore $0 \in R(A')$, $z \in R(A)$, and the proof is complete.

We continue with the following surjectivity result. Recall that $C \subset E \times E$ is said to be locally bounded if for each point $x \in cl(D(C))$ there is a neighborhood U of x such that $\cup \{Cx : x \in U\}$ is bounded.

Theorem 6. Let E be a Banach space each bounded closed convex subset of which has the fixed point property for nonexpansive mappings, and let $A \subset E \times E$ be m-accretive. If A^{-1} is locally bounded, then $R(A) = E$.

Proof. If $y_n \in Ax_n$ and $y_n \to y$, then $\{x_n\}$ is bounded because A^{-1} is bounded on a neighborhood of y. By Lemma 1, $y \in R(A)$. In other words, R(A) is closed. To see that R(A) is also open, let $y_0 \in Ax_0$, and suppose that A^{-1} is bounded on $B(y_0,R)$. Let $y \in B(y_0,R/2)$, and for positive r let x_r satisfy $y + rx_0 \in Ax_r + rx_r$. Denoting $y + rx_0 - rx_r$ by $z_r \in Ax_r$, we have $(y_0 - z_r, x_0 - x_r)_+ \geq 0$. Therefore $(y_0 - z_r, z_r - y)_+ \geq 0$, $(y_0 - y, z_r - y)_+ \geq |z_r - y|^2$, and $|z_r - y| \leq |y - y_0| < R/2$. Consequently, $|z_r - y_0| < R$ and $\{x_r\}$ is bounded. Since $\lim_{r \to 0+} z_r = y$, we see that $y \in cl(R(A)) = R(A)$. The result follows.

This theorem improves upon two results of Browder. See [4, p. 391] and [5, p. 164].

In the following corollaries we replace the local boundedness assumption by stronger hypotheses. For the Hilbert space case, see [1, p. 31]. See also [2, Theorem 3] and [3, Theorem 5].

Corollary 7. In the setting of Theorem 6, if $\lim_{\substack{x \in D(A) \\ |x| \to \infty}} \|Ax\| = \infty$, then $R(A) = E$.

Corollary 8. In the setting of Theorem 6, if there is a point $x_0 \in E$ such that $\lim_{\substack{x \in D(A) \\ |x| \to \infty}} (y, x-x_0)_+ / |x| = \infty$, where $y \in Ax$, then $R(A) = E$.

Another way to prove Corollary 7 is to observe that if A^{-1} is bounded, then for a fixed $r > 0$, $I-J_r$ is bounded on unbounded sets. Hence $\{J_r^n x\}$ is bounded, J_r has a fixed point, and the result follows. This argument also shows that if A^{-1} is bounded, then $cl(R(A)) = E$ in any Banach space [12, Theorem 1].

Corollary 8 can be proved by noting that if $0 \notin R(A)$, then by Theorem 2, $\lim_{t\to\infty} |J_t x_0| = \infty$ and $(A_t x_0, J_t x_0 - x_0)_+ < 0$. This method also provides a new sufficient condition for the existence of a zero of A.

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