Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 3, 573--590

Persistent URL: http://dml.cz/dmlcz/106022

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21,3 (1980)

CARTESIAN CLOSED FUNCTOR-STRUCTURED CATEGORIES

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Abstract: We characterize set functors F such that the functor-structured category S(F) (of pairs (A,α) where A is a set and $\alpha \subset FA$) is cartesian closed. This is so iff F covers pullbacks.

Key words: Cartesian closed category, functor-structured category, pullback preservation.

Classification: 18D15

Introduction. Functor-structured categories, introduced in [K],[HPT], are concrete categories S(F) over an arbitrary base category $\mathcal{X}$, defined via a functor $F:\mathcal{X} \longrightarrow$ Set. The objects of S(F) are pairs (A,α), where A is an object of $\mathcal{X}$ and $\alpha \subset FA$. The morphisms $f:(A,\alpha) \longrightarrow (B,\beta)$ are those morphisms in $\mathcal{X}$ for which $a \in \alpha$ implies $Ff(a) \in \beta$. These categories have a number of important properties: they are "universal" initially complete and fibre-small categories; see [AHS].

In the present paper we exhibit a necessary and sufficient condition on F in order that S(F) be cartesian closed (assuming that $\mathcal{X}$ is). The condition is in terms of the covering of pullbacks; a pullback is said to be covered by a functor if this functor maps it on a square, through which all commuting squares factorize but not necessarily uniquely. A number of examples and counterexamples is presented.

The paper is a part of a broader program of a study of concrete cartesian closed categories; see [$AK_{1,2}$].

1. Recall that a category $\mathcal{K}$ is <u>cartesian closed</u> if it has finite products and, for each object K, the induced functor

$$K\times - : \mathcal{K} \longrightarrow \mathcal{K}$$

is a left adjoint. Assuming that $\mathcal{K}$ is cocomplete and co-well-powered and has a generator then (by the dual to the special adjoint functor theorem) $\mathcal{K}$ is cartesian closed iff the functors $K\times -$ preserve coproducts and coequalizers. The last two conditions can be reformulated as follows:

(i) $\mathcal{K}$ is a <u>distributive category</u>, which means that, given objects K and L_t, $t\in T$, then the natural morphism

$$\xi : \coprod_{t\in T} (K\times L_t) \longrightarrow K \times \coprod_{t\in T} L_t$$

is an isomorphism;

(ii) $\mathcal{K}$ has <u>productive quotients</u>, which means that, given an object K and a regular epi $e:L \longrightarrow L'$ then also $1_K\times e:K\times L \longrightarrow K\times L'$ is a regular epi.

2. <u>Examples</u>. (i) The category of graphs [or binary relations (A,α), where $\alpha \subset A\times A$] and compatible maps is cartesian closed. Defining a "cartesian square functor"

$$Q: \mathrm{Set} \longrightarrow \mathrm{Set}$$

by

$$QX = X\times X \text{ and } Qf = f\times f$$

the category of graphs is the functor-structured category S(Q).

(ii) More generally, categories of relational structures are cartesian closed functor-structured categories.

(iii) The category of hypergraphs [i.e., pairs (A,α) where $\alpha \subseteq \exp A$] and compatible maps [$f:(A,\alpha) \longrightarrow (B,\beta)$ subject to $f(T) \in \beta$ for each $T \in \alpha$] is cartesian closed. This is the functor-structured category S(P) where $P:\mathrm{Set} \longrightarrow \mathrm{Set}$ is the "power-set functor " defined by

$$PX = \exp X;\ Pf = \exp f:T \longmapsto f(T).$$

3. Hypotheses. Throughout the present paper we assume that a (base) category $\mathfrak{X}$ is given such that

(i) $\mathfrak{X}$ is cocomplete, finitely complete and co-well-powered;

(ii) $\mathfrak{X}$ has a generator;

(iii) $\mathfrak{X}$ is cartesian closed, i.e., is distributive and has productive quotients.

We shall investigate functors $F:\mathfrak{X} \longrightarrow \mathrm{Set}$ with respect to the cartesian closedness of the category S(F).

While the conditions (i) and (ii) above are completely natural, the last condition excludes a number of important base-categories. Nevertheless, in case $\mathfrak{X}$ fails to be cartesian closed then so do functor-structured categories over $\mathfrak{X}$. (Since each category S(F) contains a full copy of $\mathfrak{X}$: the discrete objects $(A,\emptyset)$; this copy is closed under limits and colimits in S(F), moreover a limit of a diagram containing a discrete object is discrete and thus a "hom-object" of discrete objects is discrete.)

4. Limits and colimits in categories S(F) are naturally lifted from the base-category $\mathfrak{X}$. E.g., given objects A, B in $\mathfrak{X}$ with a product $A \times B$ (under projections π_A, π_B) then for arbitrary $\alpha \subseteq FA$ and $\beta \subseteq FB$ we have, in S(F),

$$(A,\alpha) \times (B,\beta) = (A \times B, \alpha \boxtimes \beta)$$

where

$$\alpha \boxtimes \beta = \{t \in F(A \times B);\ F\pi_A(t) \in \alpha \text{ and } F\pi_B(t) \in \beta\} ,$$

under the same projections π_A, π_B. Analogously, if A+B is a coproduct (under injections i_A, i_B) then, in S(F),

$$(A,\alpha) + (B,\beta) = (A+B, \alpha \boxplus \beta)$$

where

$$\alpha \boxplus \beta = \{t \in F(A+B);\ t \in Fi_A(\alpha) \text{ or } t \in Fi_B(\beta)\}.$$

Furthermore, if G is a generator of $\mathfrak{X}$ then $(G,\emptyset)$ is clearly a generator of S(F). Thus, the above conditions (i), (ii) on the base-category $\mathfrak{X}$ are shared by all functor-structured categories over $\mathfrak{X}$. The question of cartesian closedness thus hangs on the distributivity and the productivity of quotients in S(F).

Finally, let us remark that a morphism $f:(A,\alpha) \longrightarrow$ $\longrightarrow (B,\beta)$ in S(F) is a regular epi iff (i) f is a regular epi in $\mathfrak{X}$ and (ii) $\beta = Ff(\alpha)$.

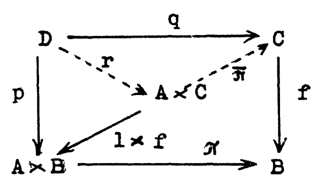
5. Projections are used abstractly below: a morphism $f:A \longrightarrow B$ is a projection if there exists an object B' such that $A = B \times B'$ under projections $f:A \longrightarrow B$ (and $f':A \longrightarrow B'$). Dual notion: injections.

6. <u>Lemma</u>. The pullback of a projection along an arbitrary morphism is a projection. I.e., given a projection π : $: A \times B \longrightarrow B$ and a morphism $f : C \longrightarrow B$, the square

$$\begin{array}{ccc} A \times C & \xrightarrow{\;\bar{\pi}\;} & C \\ {\scriptstyle 1_A \times f}\downarrow & & \downarrow{\scriptstyle f} \\ A \times B & \xrightarrow{\;\pi\;} & B \end{array}$$

is a pullback (where $\bar{\pi}$ is a projection).

Proof. Given a commuting square $\pi . p = f . q$:



define $r : D \longrightarrow A \times C$ by

$$\bar{\pi}' . r = \pi' . p \text{ and } \bar{\pi} . r = q,$$

where $\pi' : A \times B \longrightarrow A$ and $\bar{\pi}' : A \times C \longrightarrow A$ are projections. Then $p = (1 \times f) . r$ because

$$\pi' . p = \bar{\pi}' . r = \pi' . [(1 \times f) . r]$$

as well as

$$\pi . p = f . q = f . \bar{\pi} . r = \pi . [(1 \times f) . r].$$

Clearly, r is uniquely determined by $p = (1 \times f) . r$ and $q = \bar{\pi} . r$.

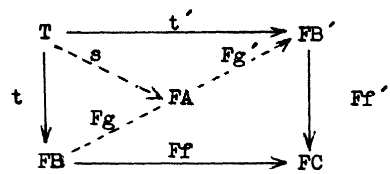
7. <u>Remark</u>. Particularly, if f is an injection $f : C \longrightarrow C + C' = B$ then we obtain a pullback of a projection and an injection:

$$
\begin{array}{ccc}
A \times C & \xrightarrow{\text{projection}} & C \\
{\scriptstyle 1_A \times \text{injection}} \downarrow & & \downarrow {\scriptstyle \text{injection}} \\
A \times (C + C') & \xrightarrow{\text{projection}} & C + C'
\end{array}
$$

8. A functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ is said to cover the pullback

$$
\begin{array}{ccc}
A' & \xrightarrow{g'} & B' \\
{\scriptstyle g} \downarrow & & \downarrow {\scriptstyle f'} \\
B & \xrightarrow{f} & C
\end{array}
$$

if for an arbitrary commuting square in $\mathcal{L}$, $Ff.t = Ff'.t'$:



there exists a morphism $s: T \longrightarrow FA$, not necessarily unique, with $t = Fg.s$ and $t' = Fg'.s$.

In case $\mathcal{L}$ = Set this means that for arbitrary points $b \in FB$ and $b' \in FB'$ subject to $Ff(b) = Ff'(b')$ there exists a point $a \in FA$ with $b = Fg(a)$ and $b' = Fg'(a')$.

9. <u>Proposition</u>. The category S(F) is distributive iff F covers each pullback of a projection and an injection.

Proof. I) Necessity.

Given a pullback as in 7:

$$\begin{array}{ccc} A\times C & \xrightarrow{\bar{\pi}} & C \\ {\scriptstyle 1_A\times i}\downarrow & & \downarrow{\scriptstyle i} \\ A\times(C+C') & \xrightarrow{\pi} & C+C' \end{array}$$

and given points

$$x\in F(A\times(C+C'));\ y\in FC$$

with $F\pi(x) = Fi(y)$,

we are to exhibit a point $z\in F(A\times C)$ subject to

$$F(1_A\times i)(z) = x \text{ and } F\bar{\pi}(z) = y.$$

Consider objects (A,FA); $(C,\{y\})$ and $(C',\emptyset)$ in $S(F)$. Clearly

(1) $\quad (A,FA)\times[(C,\{y\}) + (C',\emptyset)] = (A\times[C + C'],\alpha)$

where

$$\alpha = FA \boxtimes [\{y\} \boxplus \emptyset] = \{t\in F(A\times[C + C']);\ F\pi(t) = Fi(y)\}.$$

Thus, $x \in \alpha$. Furthermore,

(2) $\quad [(A,FA)\times(C,\{y\})] + [(A,FA)\times(C',\emptyset)] = ([A\times C] +$

$$+ [A\times C'],\beta)$$

where, denoting by $j:A\times C\longrightarrow[A\times C] + [A\times C']$ the injection, $\beta = (FA\boxtimes\{y\})\boxplus(FA\boxtimes\emptyset) = \{Fj(z);z\in F(A\times C) \text{ and } F\bar{\pi}(z) = y\}$.

By hypothesis, the isomorphism ξ (see 1(i)) is an isomorphism in $S(F)$ from the object (2) to the object (1). Hence, $x\in\alpha$ implies $F\xi^{-1}(x)\in\beta$. In other words, there exists $z\in F(A\times C)$ with $F\bar{\pi}(z) = y$ and $Fj(z) = F\xi^{-1}(x)$. Since, by the definition of ξ , we have $\xi . j = 1_A\times i$, the latter implies $x = (F\xi).Fj(z) = F(1_A\times i)(z)$.

II. Sufficiency. For arbitrary objects (B_t,β_t), $t\in T$, and (A,α) in $S(F)$ we shall prove that

$$\xi^{-1}:(A,\alpha)\times\coprod_{t\in T}(B_t,\beta_t)\longrightarrow\coprod_{t\in T}(A,\alpha)\times(B_t,\beta_t)$$

is a morphism in S(F). Then ξ is an isomorphism, since it is always a (natural) morphism. Let us denote projections by

$$\pi : A \times \coprod_{t \in T} B_t \longrightarrow A \text{ and } \bar{\pi} : A \times \coprod_{t \in T} B_t \longrightarrow \coprod_{t \in T} B_t$$

and injections by

$$i_s : B_s \longrightarrow \coprod_{t \in T} B_t \text{ and } j_s : A \times B_s \longrightarrow \coprod_{t \in T} (A \times B_t)$$

(for $s \in T$). Then

$$(A, \alpha) \times \coprod_{t \in T} (B_t, \beta_t) = (A \times \coprod_{t \in T} B_t, \gamma)$$

where a point $x \in F(A \times \coprod_{t \in T} B_t)$ fulfils

$$x \in \gamma \quad \text{iff} \quad F\pi(x) \in \alpha \quad \text{and } F\bar{\pi}(x) = Fi_s(y) \text{ for some } s \in T,\ y \in \beta_s.$$

Given such a point x we shall verify that the point $F\xi^{-1}(x)$ fulfils $F\xi^{-1}(x) = Fj_s(z)$ for some $z \in \alpha \boxtimes \beta_s$. Then, of course, ξ^{-1} is a morphism in S(F).

Put

$$B' = \coprod_{t \in T - \{s\}} B_t;$$

then we can use the covering of the pullback

$$\begin{array}{ccc} A \times B_s & \xrightarrow{\bar{\pi}_s} & B_s \\ {\scriptstyle 1_A \times i_s} \downarrow & & \downarrow {\scriptstyle i_s} \\ A \times (B_s + B') & \xrightarrow{\bar{\pi}} & B_s + B' = \coprod_{t \in T} B_t \end{array}$$

where $\bar{\pi}_s$ is the projection. Since

$$F\bar{\pi}(x) = Fi_s(y),$$

there exists $z \in F(A \times B_s)$ with

$$F(1_A \times i_s)(z) = x \text{ and } F\bar{\pi}_s(z) = y \in \beta_s.$$

The projection $\pi_s : A \times B_s \longrightarrow A$ fulfils

$$\pi_s = \pi . (1_A \times i_s),$$

hence $F\pi_s(z) = F\pi(x) \in \alpha$ as well as $F\bar{\pi}_s(z) \in \beta_s$. Hence,

$$z \in \alpha \boxtimes \beta_s.$$

And, by definition of ξ, we have $\xi . j_s = 1_A \times i_s$, therefore

$$F\xi^{-1}(x) = F\xi^{-1}.F(1_A \times i_s)(z) = Fj_s(z).$$

This concludes the proof that ξ^{-1} is a morphism.

10. Proposition. The category S(F) has productive quotients iff F covers each pullback of a projection and a regular epi.

Proof. I) Necessity. Given a pullback as in 6:

$$\begin{array}{ccc} A\times C & \xrightarrow{\ \bar{\pi}\ } & C \\ {\scriptstyle 1_A\times f}\downarrow & & \downarrow{\scriptstyle f} \\ A\times B & \xrightarrow{\ \pi\ } & B \end{array}$$

with f a regular epi and given points

$$x \in F(A\times B) \text{ and } y \in FC$$

with

$$F\pi(x) = Ff(y),$$

we are to exhibit a point $z \in F(A\times C)$ subject to

$$F(1_A \times f)(z) = x \text{ and } F\bar{\pi}(z) = y.$$

The morphism $f:(C,\{y\}) \longrightarrow (B,\{Ff(y)\})$ is a regular epimorphism in S(F) (see 4.), hence so is

$$1_A \times f:(A,FA)\times(C,\{y\}) \longrightarrow (A,FA)\times(B,\{Ff(y)\}).$$

This means that

$$FA \boxtimes \{Ff(y)\} = F(1_A \times f)(FA \boxtimes \{y\}).$$

Since $F\pi(x) = Ff(y)$, we have

$$x \in FA \boxtimes \{Ff(y)\}.$$

Hence, there exists $z \in FA \boxtimes \{y\}$ with $F(1_A \times f)(z) = x$, and, of course, $F\bar{\pi}(z) = y$.

II) Sufficiency. For each regular epi in S(F), $f: :(C, \gamma) \longrightarrow (B, \beta)$ and each object (A, α) we are to verify that

$$1_A \times f: (A, \alpha) \times (C, \gamma) \longrightarrow (A, \alpha) \times (B, \beta)$$

is a regular epi. In other words, that $\alpha \boxtimes \beta = F(1_A \times f)(\alpha \boxtimes \gamma)$. Denote projections by

$$\pi : A \times B \longrightarrow B \text{ and } \pi' : A \times B \longrightarrow A;$$

$$\bar{\pi} : A \times C \longrightarrow C \text{ and } \bar{\pi}' : A \times C \longrightarrow A.$$

For every point $x \in F(A \times B)$ with $x \in \alpha \boxtimes \beta$, i.e.,

$$F\pi(x) \in \beta \quad \text{and } F\pi'(x) \in \alpha$$

we shall find $z \in \alpha \boxtimes \gamma$ with $x = F(1_A \times f)(z)$.

Since f is a regular epi, $\beta = Ff(\gamma)$, thus, there exists $y \in FC$ with

$$F\pi(x) = Ff(y).$$

We use the covering of the following pullback

$$\begin{array}{ccc} A \times C & \xrightarrow{\bar{\pi}} & C \\ {\scriptstyle 1_A \times f}\downarrow & & \downarrow{\scriptstyle f} \\ A \times B & \xrightarrow{\pi} & B \end{array}$$

There exists $z \in F(A \times C)$ subject to $F(1_A \times f)(z) = x$ and $F\bar{\pi}(z) = y$. Since $\bar{\pi}' = \pi' . (1_A \times f)$, we have

$$F\pi'(z) = F\pi'(x) \in \alpha \quad \text{and } F\bar{\pi}(z) = y \in \beta,$$

hence $z \in \alpha \boxtimes \beta$.

11. <u>Corollary</u>. The category S(F) is cartesian closed iff F covers each pullback of a projection and a map, composed by injections and regular epis.

12. <u>Examples</u>. Every hom-functor covers (indeed, preserves) pullbacks. A product or coproduct of functors covering certain pullbacks also covers them. (On the other hand, this is not true about subfunctors or quotient functors as we shall show below.)

13. <u>Definition</u>. A category $\mathfrak{X}$ is <u>connected</u> if $\hom(A,B) \neq \emptyset$ for arbitrary objects A, B such that B is not initial.

14. <u>Theorem</u>. Let $\mathfrak{X}$ be a connected category in which each split mono is a coproduct injection. The following conditions are equivalent for each functor $F: \mathfrak{X} \longrightarrow \mathrm{Set}$, preserving finite intersections of split subobjects:

(i) S(F) is cartesian closed;

(ii) F covers pullbacks.

Proof. Assuming that F covers all pullbacks mentioned in 11., we shall prove that, in fact, F covers all pullbacks. For each morphism $f:A \longrightarrow B$ we denote by $f^* : A \longrightarrow A \times B$ the split mono defined by

$$\pi_A . f^* = 1_A \text{ and } \pi_B . f^* = f.$$

And for each pair of morphisms $f:A \longrightarrow B$ and $g:A \longrightarrow C$ we denote by $f \dot{\times} g : A \longrightarrow B \times C$ the morphism ,defined by

$$\pi_B . (f \dot{\times} g) = f \text{ and } \pi_C . (f \dot{\times} g) = g.$$

(Thus $f^* = 1_A \dot{\times} f$.)

Given a pullback

$$\begin{array}{ccc} C & \xrightarrow{\quad g_2 \quad} & A_2 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ A_1 & \xrightarrow{\quad f_1 \quad} & B \end{array}$$

each of the following four squares is a pullback, too:

$$\begin{array}{ccccc}
C & \xrightarrow{g_1 \dot{\times} g_2} & A_1 \times A_2 & \xrightarrow{\text{projection}} & A_2 \\
\downarrow{\scriptstyle g_1 \dot{\times} g_2} & & \downarrow{\scriptstyle 1_{A_1} \times f_2} & & \downarrow{\scriptstyle f_2^*} \\
A_1 \times A_2 & \xrightarrow{f_1^* \times 1_{A_2}} & A_1 \times B \times A_2 & \xrightarrow{\text{projection}} & A_2 \times B \\
\downarrow{\scriptstyle \text{projection}} & & \downarrow{\scriptstyle \text{projection}} & & \downarrow{\scriptstyle \text{projection}} \\
A_1 & \xrightarrow{f_1^*} & A_1 \times B & \xrightarrow{\text{projection}} & B
\end{array}$$

It is clear that if F covers each of these four pullbacks then F covers the pullback of f_1 and f_2. Since $f_1^* \times 1_{A_2}$ and $1_{A_1} \times f_2^*$ are split monos, F preserves their pullback (= intersection) by hypothesis. And F covers the two adjacent pullbacks since S(F) is a distributive category and f_1^*, f_2^* are split monos, hence injections. Thus, it remains to prove that F covers the pullback of two projections down to the right.

(a) Let there exist a morphism from B to A_1 or A_2. Say,

$$p : B \longrightarrow A_1 .$$

Then the projection $\pi_B : A \times B \longrightarrow B$ is a split epi, since we have $\pi_B \cdot (p \dot{\times} 1_B) = 1_B$. Since S(F) has productive quotients, F covers the pullback in question.

(b) Let there be no morphism from B to A_1 nor A_2. Then both A_1 and A_2 are initial objects and, moreover, for each non-initial object X we have $\hom(X,A_1) = \emptyset = \hom(X,A_2)$. (Indeed, since $\mathcal{X}$ is connected, we have $\hom(B,X) \neq \emptyset$ for each non-initial X !) Thus, in the original pullback of f_1 and f_2, both g_1 and g_2 are isomorphisms, hence the pullback is covered.

15. Corollary. A set-functor

$$F: Set \longrightarrow Set$$

has the property that S(F) is cartesian closed iff F covers non-empty pullbacks.

Proof. The category $\mathcal{X}$ = Set is connected, every split mono is a coproduct injection and the hypotheses 3. above hold. For each functor F there exists a functor F', which preserves finite intersections and coincides with F on all non-void sets (and maps). See [T_1]. It is easy to see that S(F) is cartesian closed iff so is $S(F')$.

16. Examples of set functors.

(a) All hom-functors, the power set functor (see 2(iii)) and all compositions, products and coproducts of these, cover pullbacks.

(b) The first example of S(F) not cartesian closed is due to Jiří Vinárek. Here F is the following quotient functor of the cartesian square functor Q (see 2(i)):

$$FX = X \times X/\sim$$

where

$$(x_1,x_2) \sim (x_1',x_2') \text{ iff either } (x_1,x_2) = (x_1',x_2')$$
$$\text{or } x_1 = x_2 \text{ and } x_1' = x_2'.$$

On maps $f: X \longrightarrow Y$, denoting by [ ] the equivalence classes:

$$Ff[(x_1,x_2)] = [f(x_1),f(x_2)].$$

Then S(F) can be viewed as the full subcategory of the category S(Q) of graphs over all reflexive and all antireflexive graphs (i.e. graphs (A,α) such that, if one loop (a,a) is in α, then all loops are).

This category is not distributive. Consider graphs

(B_1, β_1): $\overset{\bullet}{b_1}$ $\quad$ (B_2, β_2): $\overset{\circlearrowleft}{b_2}$ $\quad$ (A, α): $\overset{\bullet}{a_1} \longrightarrow \overset{\bullet}{a_2}$

Then $(A,\alpha) \times [(B_1, \beta_1) + (B_2, \beta_2)]$ is the following graph

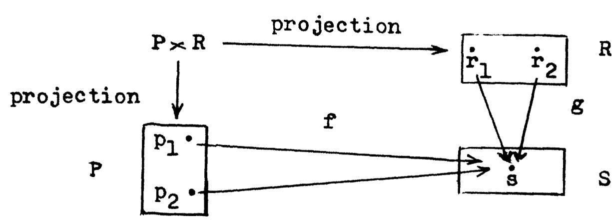
while $[(A,\alpha) \times (B_1, \beta_1)] + [(A,\alpha) \times (B_2, \beta_2)]$ is the following graph

(c) Example of a subfunctor $Q_{2,3}$ of the hom-functor hom$(A,-)$ with card $A = 3$ such that $Q_{2,3}$ does not cover pullbacks:

$Q_{2,3}X = \{(x,y,z) \in X \times X \times X;\ \text{card}\,\{x,y,z\} \leq 2\}$;

$Q_{2,3}f(x,y,z) = (f(x), f(y), f(z))$.

Consider the pullback



The points $(p_1,p_1,p_2) \in Q_{2,3}P$ and $(r_1,r_2,r_2) \in Q_{2,3}R$ fulfil

$$Q_{2,3}f(p_1,p_1,p_2) = Q_{2,3}g(r_1,r_2,r_2),$$

yet there exists no $(x,y,z) \in Q_{2,3}(P \times R)$ which the projections would map to the given points.

(d) S(F) need not be cartesian closed even if it is distributive. Let F be the set functor, obtained by merging two copies of P in the singleton-set subfunctor: on objects

$$FX = (\exp X - \{\{x\};\ x \in X\}) \times \{1,2\} \cup \{\{x\};\ x \in X\};$$

on morphisms $f: X \longrightarrow Y$;

$Ff(T,i) = (f(T),i)$ for $i=1,2$ and $T \in X$ with card $f(T) \neq 1$;

$Ff(T,i) = f(T)$ if card $f(T) = 1$

$Ff(\{x\}) = \{f(x)\}$.

Then F is easily seen to preserve preimages (pullbacks of a morphism and a mono), hence S(F) is a distributive category (see 9.). Yet, F does not cover the pullback of (c): for the points

$$P \times \{1\} \in FP \text{ and } R \times \{2\} \in FR$$

there exists no corresponding point in $F(P \times R)$.

(e) S(F) need not be cartesian closed even if it has productive quotients. The following example is due to V. Trnková.

Let us define a quotient functor F of the power-set functor P:

$$FX = (\exp X)/\sim$$

where $A \sim B$ means that the symmetric difference $(A-B) \cup (B-A)$ is finite;

$Ff[A] = [f(A)]$ for each map $f: X \longrightarrow Y$ and each $A \subset X$.

This functor covers pullbacks of surjections. Indeed, consider such a pullback:

$$\begin{array}{ccc} T & \xrightarrow{\;f'\;} & R \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ P & \xrightarrow[\;f\;]{} & S \end{array}$$

Let $A \subset P$ and $B \subset R$ be subsets with

$$Ff[A] = Fg[B];$$

then the symmetric difference of $f(A)$ and $g(B)$ is finite. For each point $s \in f(A) - g(B)$ choose a point $b_s \in R$ with

$$g(b_s) = s$$

and put

$$B_1 = B \cup \{b_s;\ s \in f(A) - g(B)\}.$$

Then

$$B_1 \sim B \text{ and } g(B_1) = f(A) \cup g(B).$$

Analogously we find a set $A_1 \subset P$ subject to

$$A_1 \sim A \text{ and } f(A_1) = f(A) \cup g(B).$$

Since $f(A_1) = g(B_1)$, there clearly exists a set $C \subset T$ with $f'(C) = B_1$ and $g'(C) = A_1$. Then the point $[C] \in FT$ fulfils

$$Ff'[C] = [B_1] = [B];\ Fg'[C] = [A_1] = [A].$$

On the other hand, F fails to cover e.g. the pullback of the characteristic function $f\colon \omega_0 \longrightarrow \{0,1\}$ of the set of all even numbers, and the inclusion map $g\colon \{0\} \longrightarrow \{0,1\}$.

(f) Let F be a super-finitary functor, i.e., there exists a finite set M such that for each set X we have

$$FX = \bigcup_{f: M \to X} Ff(FM).$$

Then F covers pullbacks iff F is isomorphic to a finite coproduct of functors

$$\hom(A,-)/G$$

where A is a finite set, G is a permutation group on A and hom(A,-)/G is the quotient functor of hom(A,-), where two maps $f, g \in hom(A,X)$ are identified iff

$$f = g.\pi \quad \text{for some permutation } \pi \in G .$$

See [T_2].

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(Oblatum 6.3. 1980)