Commentationes Mathematicae Universitatis Carolinae

Sin Min Lee

On a simple one-element extension of left zero semigroups

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 313--318

Persistent URL: http://dml.cz/dmlcz/105998

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,2 (1980)

ON A SIMPLE ONE-ELEMENT EXTENSION OF LEFT ZERO SEMIGROUPS

LEE SIN-MIN

Abstract: For each finite left zero semigroup L_n of order n, we embedded it into a simple groupoid S_n of order n+1. We show that S_n is rigid if $n \geq 3$. It is shown that the variety of groupoids generated by S_2 contains infinitely many finite non-isomorphic simple groupoids such that each of them generates the same variety. This provides a solution to Problem 67 of Birkhoff [1].

Key words: Left zero semigroups, one-element extension, simple groupoids, residually small variety.

Classification: 08A05

---

§ 1. Introduction. A groupoid $\langle G'; \circ \rangle$ is said to be an extension of another groupoid $\langle G; \circ \rangle$ if G is isomorphic to a subgroupoid of G'. We identify G with the subgroupoid of G'. If G' is simple, i.e. its lattice of congruences is the two-element lattice, then we say G' is a simple extension of G.

In [3], we show that any finite or countable groupoid G has a simple extension G' such that $|G'-G| = 1$. We call G' a simple one-element extension of G. In this paper we want to introduce another simple one-element extension for each finite left zero semigroup, i.e. the semigroup satisfies the identity $x \circ y = x$. It is well known that any left zero semigroup of order greater than two is not simple and has a large group of automorphisms. It is shown that the simple one-element extension S_n of the left zero semigroup L_n of order $n \geq 3$ has a trivial group of automorphisms.

We show that the variety $Var(S_2)$ of groupoids generated by S_2 has infinitely many non-isomorphic simple groupoids such that each of them generates the whole variety. This provides a solution to the problem which is raised by B. Jonsson in Birkhoff's book [1].

§ 2. The simple one-element extension of finite left zero semigroups. Let $\mathbb{N}$ be the set of all natural numbers. Denote by $\mathbb{N}^*$ the set union of $\mathbb{N}$ and a symbol e not in $\mathbb{N}$. We define the binary operation $\circ$ on $\mathbb{N}^*$ as follows:

(1) $x \circ x = x$ for all x in $\mathbb{N}^*$,

(2) $x \circ e = 1$ for all x in $\mathbb{N}$,

(3) $x \circ y = x$ for all x, y in $\mathbb{N}$,

(4) $e \circ x = \begin{cases} e \text{ if } x=1 \\ x-1 \text{ if } x \in \mathbb{N} - \{1\}. \end{cases}$

The groupoid $\langle \mathbb{N}^* ; \circ \rangle$ is an idempotent groupoid which contains a countable left zero semigroup $\langle \mathbb{N} ; \circ \rangle$. For each $n \geq 1$, we denote by L_n (respectively S_n) the subgroupoid $\{1,2,\ldots,n\}$ (respectively $\{e\} \cup L_n$) of $\mathbb{N}^*$. It is obvious that L_2 is isomorphic to S_1 and L_n is a subgroupoid of S_n.

Theorem 2.1. The groupoid S_n is a simple one-element extension of L_n.

Proof. Let Θ be a non-identity congruence of S_n. If $e\Theta m$ for $m \in L_n$ then $e \circ e\ \Theta\ e \circ m$, i.e. $e\ \Theta(m-1)$. If we left multiply both sides of the congruence by e successively, we will reach $e\ \Theta\ 1$. Then for each $x \in L_n$ we have $x \circ e\ \Theta\ x \circ 1$ i.e. $1\ \Theta\ x$. Hence by the transitivity of Θ we conclude that $\Theta = S_n \times S_n$. If we have $x\ \Theta\ y$ where x, y in L_n and $x < y$ then left multiplying both sides of the congruence by e successively x-1 times, we obtain $e\ \Theta\ (y-x+1)$ which implies $\Theta =$ $= S_n \times S_n$ by the above result. Hence S_n is simple.

Corollary 2.2. The groupoid $\langle \mathbb{N}^*; \circ \rangle$ is simple.

The group of automorphisms of S_1 is the cyclic group of order two. The groupoid S_2 has a non-trivial automorphism f which maps e to 2, 2 to e and 1 to 1. We recall that a groupoid G is said to be rigid if its group Aut(G) of automorphisms is trivial.

Theorem 2.3. The groupoid S_n is rigid if and only if $n \geq 3$.

Proof. We assume $n \geq 3$ and f is an automorphism of L_n.

Claim: f(e)=e.

If f(e)=i where $i \in L_n$ then there exists $j \in L_n$ such that f(j)=e. Since $n \geq 3$, we can find two elements s, t distinct from j. Hence $f(s) \neq e \neq f(t)$. As $j \circ s = j \circ t = j$, we obtain $f(j \circ s)=f(j \circ t)=e$ i.e. $e \circ f(s)=e \circ f(t)=e$. Hence $f(s)=f(t)=1$, a contradiction. Therefore we must have f(e)=e.

Now $f(1)=f(x \circ e)=f(x) \circ f(e)=f(x) \circ e = 1$ and by induction we can show that f(k)=k for any $k \in L_n$. Hence f is the identity map. Thus S_n is rigid.

By the same argument we have

Corollary 2.4. The groupoid $\langle \mathbb{N}^*; \circ \rangle$ is rigid.

§ 3. Simple groupoids in the variety generated by S_2. In this section we show that the variety of groupoids $Var(S_2)$ which is generated by S_2 has arbitrarily large simple groupoids. This provides an example of a locally finite variety of algebras which is not residually small.

For each non-empty set X, we denote by $X^+ = X \cup \{1\}$ where $1 \notin X$. We define a binary operation $\circ : X^+ \times X^+ \longrightarrow X^+$ as follows:

(1) $x \circ x = x$ for any $x \in X^+$,

(2) $x \circ y = \begin{cases} 1 & \text{if } x \neq y \text{ in } X \text{ or } x=1,\ y \in X \\ x & \text{otherwise.} \end{cases}$

Theorem 3.5. The groupoid $\langle X^+; \circ \rangle$ is simple.

Proof. If $|X| = 1$ then X^+ is isomorphic to the groupoid L_2 which is simple.

If $|X| \geq 2$ and θ is a non-identity congruence of X^+ we want to show that $\theta = X^+ \times X^+$. If $1\ \theta\ x$ where $x \in X$ then left multiplying both sides of the congruence by $y \in X - \{x\}$ we obtain $y\ \theta\ 1$. Thus θ is the universal congruence. If $x\ \theta\ y$ where x, y in X then $x \circ x\ \theta\ x \circ y$ would imply $x\ \theta\ 1$ which reduces to the previous case. Therefore $\langle X^+; \circ \rangle$ is simple.

Theorem 3.6. The groupoid X^+ is in $Var(S_2)$.

Proof. Let S_2^X be the direct power of S_2. It is clear that S_2^X is in $Var(S_2)$. For each $x \in X$, let $\underset{\sim}{x}$ be the map from X to S_2 such that

$$\underset{\sim}{x}(y) = \begin{cases} 2 & \text{if } y=x \\ e & \text{otherwise} \end{cases}$$

Let $\underset{\sim}{1}:X \longrightarrow S_2$ be the constant map $\underset{\sim}{1}(y)=1$ for all y in X.

Let P(X) be the subgroupoid of S_2^X generated by $\{\underset{\sim}{x}:x \in X\} \cup \{\underset{\sim}{1}\}$. Then $P(X)-\{\underset{\sim}{x}:x \in X\}$ contains only maps α such that $\alpha(y)$ is either 1 or e. As $\{1,e\}$ is a left zero semigroup then $P(X)-\{\underset{\sim}{x}:x \in X\}$ is a subgroupoid of P(X).

We consider the relation Θ defined on P(X) by setting $\alpha \Theta \beta$ if and only if either $\alpha = \beta$ or $\alpha, \beta \in P(X)-\{\underset{\sim}{x}:x \in X\}$. We shall denote the equivalence class containing α by $[\alpha]\Theta$. It is obvious that Θ is a congruence relation and $[\underset{\sim}{1}]\Theta = P(X)-\{\underset{\sim}{x}:x \in X\}$.

The map $\Phi : X^+ \longrightarrow P(X)/\Theta$ defined by $\Phi(x) = [\underset{\sim}{x}]\Theta$ and $\Phi(1) = [\underset{\sim}{1}]\Theta$ is an isomorphism. Thus X^+ is in $Var(S_2)$.

<u>Theorem 3.7</u>. For any set X with cardinality greater than one we have $Var(X^+) = Var(S_2)$.

Proof. Let $x,y \in X$ then we see that the subgroupoid $\{1,x,y\}$ of X^+ has the following Cayley table:

$\circ$	x	1	y
x	x	x	1
1	1	1	1
y	1	y	y

The above groupoid is isomorphic to S_2 under the homomorphism $f:x \longrightarrow e$, $1 \longrightarrow 1$ and $y \longrightarrow 2$. Thus $S_2 \in Var(X^+)$. With Theorem 3.6 we conclude that $Var(X^+) = Var(S_2)$.

The above result shows that in $Var(S_2)$ there exist infinitely many non-isomorphic simple groupoids each of which generates $Var(S_2)$. This gives a solution to the Problem 67 in Birkhoff's book [1].

R e f e r e n c e s

[1] G. BIRKHOFF: Lattice theory, Amer. Math. Soc. Colloq. Publ. No. 25, 3rd Edition (Providence, 1967).

[2] A.H. CLIFFORD and G.B. PRESTON: The algebraic theory of semigroups, vol. 1, Amer. Math. Soc., Providence, 1962.

[3] LEE SIN-MIN: On a simple one-element extension of countable and finite n-groupoids (to appear).

[4] LEE SIN-MIN and SEIN-AYE: On simple groupoids, Nanta Math. 8(1975), 30-33.

[5] W. TAYLOR: Residually small varieties, Algebra Universalis 2(1972), 33-53.

Université du Paris-Sud
Bâtiment 425
91405 Orsay
France

University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

(Oblatum 21.9. 1979)