Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 247--261

Persistent URL: http://dml.cz/dmlcz/105993

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21,2 (1980)

ON BICOMPACTA WHICH ARE UNIONS OF SPACES DEFINED BY MEANS OF COVERINGS

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Abstract: Let X be a bicompact space which is the union of infinitely many subspaces of a class $\mathcal{P}$, defined by means of coverings: Lindelöf, metalindelöf, developable, weakly-$\delta\theta$-refinable etc. What can be said about the sequentiality of X, about the existence of a G_δ-point in X ? We study this problem and receive some results which are applied to the investigation of bicompact subspaces of some unions of Σ-products of metric spaces.

Key words: Bicompact spaces, sequential spaces, G_δ-point metalindelöf spaces, weakly-$\delta\theta$-refinable spaces.

Classification: 54D30

Let $\mathcal{P}$ be a class of spaces, defined by means of coverings. In this note we consider the following problem: if a bicompact Hausdorff space is the union of a certain family of spaces which are the elements of $\mathcal{P}$, what can be said about the existence of G_δ-points and about the sequentiality of this bicompactum?

In special cases, this question was investigated by A.V. Arhangel'skii [1],[2],[3] and some other authors [4], [5]. In this note we considerably strengthen the results of the papers and [3],[5], and solve some problems from [3]. Our interest in the bicompacta which are the unions of spaces, defined by means of coverings is stimulated also by the fact that every bicompactum which is embedded in Σ-products of real lines, is hereditarily metalindelöf.

We think that one of the main corollaries of this note is that the existence of a dense set of G_δ-points in a bicompact Hausdorff space very often implies the sequentiality of this space.

We adopt the terminology of [6]. The space X is called metalindelöf if every open covering of X can be refined by an open point-countable covering [7].

The space X is called weakly-$\delta\theta$-refinable [8] if every open covering of X can be refined by an open covering $\mathcal{V} = \cup \mathcal{V}_n$ such that for every $x \in X$ there is such a natural n that X belongs to at most countably many elements of $\mathcal{V}_n$.

The class of weakly-$\delta\theta$-refinable spaces includes all metric σ-metrizable, paracompact, developable, metalindelöf and other classes of spaces, defined by means of coverings. In this class, the countable compactness is equivalent to bicompactness [8].

If $\mathcal{P}$ is a certain property of a space, then we say that a space X is a pointly-$\mathcal{P}$-space, if for every $x \in X$ the subspace $X \setminus x$ has the property $\mathcal{P}$. Note that the property of being pointly-$\mathcal{P}$ is weaker than the hereditarily $\mathcal{P}$-property.

Now, if τ is a topology on X, then τ_λ (where λ is an infinite cardinal) denotes the λ-modification of τ [6] (i.e. such a topology on X that the family of all sets which are the intersections of λ many open in τ sets, is a base of this topology).

Σ_*-product of metric spaces X_α with a basic point

(x_α) is a subspace of a product $\prod X_\alpha$ such that for every $\varepsilon > 0$ and for every $(y_\alpha) \in \Sigma_*$, $|\{\alpha : \rho(y_\alpha, x_\alpha) > \varepsilon\}| < \aleph_0$ [9].

As usually, a Σ-product (σ-product) of spaces X_α with a basic point (x_α) is a subspace of a product $\prod X_\alpha$, such that for every $(y_\alpha) \in \Sigma\ (\sigma)\ |\{\alpha : y_\alpha \neq x_\alpha\}| \leq \aleph_0$ $(< \aleph_0)$.

A space is called τ-monolithic [12] iff for every A $|A| \leq \tau$ it follows that $nw([A]) \leq \tau$.

1. G_δ-points and non-trivial converging sequences

We begin with the following

Definition 1. A point x_0 is called a super Fréchet point, if for every $A \subseteq X$ such that $x_0 \in [A]$ and λ - the first cardinal such that $x \in [A]_{\tau_\lambda}$ there exists an Alexandrov supersequence $S \subseteq A$ such that $|S| = \lambda$ and S converges to x_0 (i.e. $S \cup x_0$ is a one-point compactification of S).

We also name the space a super-Fréchet space, iff each point $x_0 \in X$ is a super-Fréchet point.

Obviously, the super-Fréchet property implies the Fréchet-Uryson property.

Proposition 1. If X is a bicompactum, $x_0 \in X$, and $X \setminus \{x_0\}$ is a metalindelöf space, then x_0 is a super-Fréchet point.

Proof: Let $x_0 \in [A]$ and $\psi(x_0, A) = \lambda$. Let γ be a point-countable covering of $Y = [A] \setminus \{x_0\}$ by open sets, such that $[U] \not\ni x_0$ for every $U \in \gamma$.

Suppose, first, that $\lambda = \aleph_0$. For each $x \in Y$ let us index the elements of γ, containing x as $\{U_1(x), U_2(x), \ldots$

$\ldots, U_k(x), \ldots\}$ and let $\gamma_n(x) = \bigcup_{k=1}^{m} \{U_k(x)\}$. Let $x_1 \in A$, and for every natural n choose $x_n \in A \setminus \bigcup_{k=1}^{n-1} \gamma_{n-1}(x_k)$.
($A \setminus \bigcup_{k=1}^{n-1} \gamma_{n-1}(x_k) \neq \emptyset$, otherwise $x_0 \notin [A]$). The set $\{x_n\}$ is discrete in Y. Really, let $z \in Y$ and $U \in \gamma$ such that $U \ni z$. Now, if $U \ni x_n$ for some n, then $U = U_k(x_n)$ for some k and so $x_m \notin U$ for every $m \geq \max\{k,n\}$. It follows that $x_n \longrightarrow x_0$, because $[A]$ is a bicompactum.

Suppose that $\lambda > \aleph_0$. Let $y_0 \in A$ and for every $\alpha < \Omega(\lambda)$ choose $y_\alpha \in A \setminus \cup\{\gamma(y_\beta) : \beta < \alpha\}$. ($A \setminus \cup\{\gamma(y_\beta) : \beta < \alpha\} \neq \emptyset$, otherwise $\psi(x_0, A) < \lambda$). The set $\{y_\alpha : \alpha < \Omega(\lambda)$ is obviously discrete in Y and $|\{y_\alpha : \alpha < \Omega(\lambda)\}| = \lambda$. It follows that $y_\alpha \longrightarrow x_0$, because $[A]$ is a bicompactum.

Proposition 2. Let X be a pointly-metalindelöf bicompactum, then X is Fréchet-Uryson and a set of G_δ-points is dense in X.

Proof: X is a Fréchet-Uryson according to Proposition 1. Then according to one lemma of A.V. Arhangel'skii [6], there exists a countable $S \subseteq X$ and a bicompact $F \subseteq X$ which is G_δ in X such that $[S] \supseteq F$. Let $x_0 \in F$, then $[S] \setminus \{x\} = Y$ is a metalindelöf space, but Y is separable, therefore Y is Lindelöf and this implies that x_0 is a G_δ-point in $[S]$. It follows that x_0 is a G_δ-point in F and hence in X.

Proposition 3. Let X be a bicompactum, $t(X) \leq \aleph_0$, $X = \cup\{X_\alpha : \alpha < \omega_1\}$ and for each α

1. if $A \subseteq X_\alpha$ and A is countable, then $[A]_{X_\alpha}$ is Lindelöf,

2. if $F \subseteq X_\alpha$ and F is a bicompactum, then F contains a G_δ-point (in F),

then X also contains a G_δ-point.

Proof: On the contrary, suppose X does not contain any G_δ-point, then every G_δ-bicompactum F in X also does not contain any G_δ-point. Suppose that $\beta < \omega_1$ and for each $\alpha < \beta$ we have already defined a family of bicompact $\{F_\alpha\}$ with the following conditions:

1) $F_{\alpha'} \subseteq F_{\alpha''}$ if $\alpha' > \alpha''$,

2) F_α is a G_δ-bicompactum in X,

3) $F_\alpha \cap X_\alpha = \emptyset$.

Let us construct F_β with the same properties. Let $F^0_\beta = \bigcap\{F_\alpha : \alpha < \beta\}$. Then F^0_β is a G_δ-set in X. If $F^0_\beta \cap X_\beta \neq \emptyset$, then let x_1 be an arbitrary point of $F^0_\beta \cap X_\beta$ and K_1 be an arbitrary G_δ-bicompactum in F^0_β, containing x_1. Suppose $j < \omega_1$ and for each $\alpha < j$ we have already constructed a family of points $\{x_\alpha\}$ and bicompacta K_α such that:

a) $x_\alpha \in K_\alpha \cap X_\beta$,

b) $[\{x_{\alpha'} : \alpha' < \alpha\}] \cap K_\alpha = \emptyset$,

c) $K_{\alpha'} \subseteq K_{\alpha''}$ if $\alpha' > \alpha''$,

d) K_α is a G_δ-bicompactum in F^0_β.

Let $K^0_j = \bigcap\{K_\alpha : \alpha < j\}$. It is a G_δ-bicompactum in F^0_β. There are two possibilities:

I. $[\{x_\alpha : \alpha' < j\}] \supset K^0_j \cap X_\beta$,

II. there exists $x_j \in (K^0_j \cap X_\beta) \setminus [\bigcup\{x_\alpha : \alpha < j\}]$. Then let K_j be an arbitrary G_δ-bicompactum, containing x_j and contained in $K^0_j \setminus [\{x_\alpha : \alpha < j\}]$ (it is possible because of the condition 1. of our proposition). It is clear that a) - d) are fulfilled.

If for every $j < \omega_1$ we always have the possibility II, then we have a free sequence $\{x_j\}_{j<\omega_1}$ in a bicompactum of countable tightness. That is impossible [6], therefore there is $j_0 < \omega_1$ such that $[\{x_\alpha : \alpha < j_0\}] \supset K_j^0 \cap X_\beta$. If $K_j^0 \cap \cap X_\beta = \emptyset$, let $F_\beta = K_j^0$. But if $K_j^0 \cap X_\beta \neq \emptyset$, then this space is Lindelöf, because $[\{x_\alpha : \alpha < j_0\}] \cap X_\beta = [\{x_\alpha : : \alpha < j_0\}]_{X_\beta}$ and because of the first condition of our proposition.

K_j^0 is a G_δ-bicompactum in X, therefore K_j^0 does not contain any G_δ-point and therefore $K_j^0 \not\subset X_\beta$, so there exists a G_δ-bicompactum $K \subset K_j^0$ such that $K \cap X_\beta = \emptyset$ (here we use the fact that $K_j^0 \cap X_\beta$ is Lindelöf). Let $F_\beta = K$. Obviously, the conditions 1) - 3) are satisfied.

$\{F_\alpha : \alpha < \omega_1\}$ is a decreasing sequence of bicompacta. But then $\cap\{F_\alpha : \alpha < \omega_1\} \neq \emptyset$, and that is impossible, because of the condition 3) together with $X = \cup\{X_\alpha : \alpha < \omega_1\}$.

<u>Corollary 1</u>. Let $X = \cup\{X_\alpha : \alpha < \omega_1\}$ and X be a bicompactum of countable tightness, then each of the following conditions implies the existence of a dense set of G_δ-points in X:

a) for every α, X_α is pointly-metalindelöf;

b) $(2^{\aleph_1} > 2^{\aleph_0})$ for every α, X_α is metalindelöf and sequential,

c) for every α, X_α is embedded in some Σ-product of separable metric spaces,

d) for every α, X_α is $\aleph_0$-monolithic and $t(X_\alpha) \leq \leq \aleph_0$,

e) for every α, X_α is a space with closure-preserving covering of compact sets.

In view of Proposition 3 we can arise a problem: is the proposition 3 true without the condition $t(X) \leq \aleph_0$? (or may be some points of Corollary 1?)

We have obtained some partial results in this way:

Proposition 4. Let X be a bicompactum, $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$ and for every α,

1. X_α is Lindelöf,

2. if $F \subseteq X_\alpha$ and F is a bicompactum, then F contains a G_δ-point (in F),

then X also contains a G_δ-point.

Proof: Suppose it is not true. Then as in the proof of Proposition 3 we may define for every $\alpha < \beta$ a family of bicompacta $\{F_\alpha\}$ answering the requirements 1) – 3) of that Proposition. If $F_\beta^\circ = \bigcap\{F_\alpha : \alpha < \beta\}$, then F_β° is a G_δ-bicompactum. Therefore $F_\beta^\circ \nsubseteq X_\beta$ (otherwise it contains a G_δ-point). Let $y \in F_\beta^\circ \setminus X_\beta$. $F_\beta^\circ \cap X_\beta$ is a Lindelöf space, so there exists a G_δ-bicompactum $B(y) \ni y$ such that $B(y) \cap (F_\beta^\circ \cap \cap X_\beta) = \emptyset$. Then $F_\beta = F_\beta^\circ \cap B(y)$ also answer the requirements 1) – 3). It is clear that $\bigcap\{F_\beta : \beta < \omega_1\}$, and we again have the contradiction in view of 3).

Corollary 2. Let $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$ and X be a bicompactum. Then each of the following conditions implies the existence of a dense set of G_δ-points in X,

a) for every α, X_α is pointly-Lindelöf,

b) $(2^{\aleph_1} > 2^{\aleph_0})$ for every α, X_α is Lindelöf and sequential;

c) for every α, X_α is embedded in some σ-product of separable metric spaces.

Remark. Parts c), d) and e) of Corollary 1 and part c) of Corollary 2 are the essential generalization of the corresponding properties of Eberlein, Corson and monolithic bicompacta of countable tightness.

Proposition 5. Let X be a bicompactum, $X = \bigcup \{ X_\alpha : \alpha < \omega_1 \}$, and for each α

1. if $A \subseteq X_\alpha$ and A is countable, then $[A]_{X_\alpha}$ is Lindelöf,

2. if $F \subseteq X_\alpha$ and F is an infinite bicompactum, then F contains a non-trivial converging sequence,

then X also contains a non-trivial converging sequence.

Proof: Suppose, on the contrary, that X does not contain a non-trivial converging sequence.
Suppose $\beta < \omega_1$ and for each $\alpha < \beta$ we have already defined a family of bicompacta $\{ F_\alpha \}$ with the following conditions:

1) $F_{\alpha'} \subseteq F_{\alpha''}$ if $\alpha' > \alpha''$,

2) F_α is infinite,

3) $F_\alpha \cap X_\alpha = \emptyset$.

We shall construct F_β with the same properties. Let $F^0_\beta = \bigcap \{ F_\alpha : \alpha < \beta \}$. If β is a non-limit ordinal, then F^0_β is infinite according to 2). Now, let β be a limit ordinal and F^0_β be finite, then if $\beta = \lim_{m \to \infty} \alpha_n$ and $x_n \in F_{\alpha+1} \setminus F_\alpha$, then $[\{x_n\}] \setminus \{x_n\} \subseteq F^0_\beta$ and is also finite, but it means that $[\{x_n\}]$ is a countable metrizable compactum, and hence contains a non-trivial converging sequence and that is impossible, therefore F^0_β is infinite.

I. If $F^0_\beta \cap X_\beta$ is finite, then $F^0_\beta \setminus X_\beta$ is infinite, therefore there is an infinite bicompacum $F_\beta \subseteq F^0_\beta$ such that $F_\beta \cap \cap X_\beta = \emptyset$.

II. If $F^0_\beta \cap X_\beta$ is infinite, then it is an infinite closed set in X_β. Let S be a countable subset of $F^0_\beta \cap X_\beta$, then $[S] \subseteq F^0_\beta$ and $[S] \setminus X_\beta \ni \{y\}$, because otherwise $[S] \subseteq X_\beta$ and $[S]$ contains a non-trivival converging sequence according to the conditions of our proposition. The same arguments make us sure that $\{y\}$ may be considered as a non-isolated point of $[S]$. Besides, $[S]_{X_\beta} = [S] \cap X_\beta$ and hence is a Lindelöf space. Therefore, there exist a G_δ in $[S]$ bicompactum $B(y) \ni y$, contained in $[S]$, and a countable covering $\{U_i\}$ of $[S] \cap X_\beta$ such that $B(y) \cap (\bigcup_{i=1}^{\infty} U_i) = \emptyset$, and therefore $B(y) \cap X_\beta = \emptyset$. It is clear that $B(y)$ is infinite (otherwise $\{y\}$ is a non-isolated G_δ-point in $[S]$) and so we can define $F_\beta = B(y)$. Obviously the conditions 1) - 3) are now fulfilled. But then according to 1) $\bigcap \{F_\beta : \beta < \omega_1\} = \emptyset$ and that is impossible according to 3).

Corollary 3. Let X be a bicompactum, $X = \bigcup \{X_\alpha : \alpha < \omega_1\}$ and one of the following conditions be fulfilled:

1. for every α, X_α is pointly-metalindelöf,
2. for every α, X_α is $\aleph_0$-monolithic and $t(X_\alpha) \leq \aleph_0$,

then X contains a non-trivial converging sequence.

2. CC-closed spaces and sequential spaces

In our following arguments, the next notion will play a key role.

Definition 2. We shall call a space countably compact closed (briefly CC-closed) if every countably compact subspace of X is closed in X.

The class of CC-closed spaces obviously contains all T_1 sequential spaces, but also some others, far from sequential spaces, for example, all T_1 spaces, in which countably compact sets are finite.

We shall start with the following

Lemma 1. Let X be a Hausdorff space, $x_0 \in X$, and $X \setminus \{x_0\}$

is a weakly-$\sigma\theta$-refinable space, then for each countably compact $A \subseteq X \setminus \{x_0\}$ always $[A] \subseteq X \setminus \{x_0\}$.

<u>Proof</u>: Let A be countably compact and $A \subseteq X \setminus \{x_0\}$. Let $\mathcal{U} = \{U(x)$ such that $[U(x)] \not\ni x_0\}$. Let $\mathcal{V}$ be a weakly-$\sigma\theta$-refining of $\mathcal{U}$. Then according to [8] we can find a finite subfamily of $\mathcal{V}$ (denote $\{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$), which covers a countably compact set A. Now we have $[A] \subseteq \bigcup_{i=1}^{n} [V_i] \subseteq \bigcup \{[U(x)] : U(x) \in \{\mathcal{U}\} \subseteq X \setminus \{x_0\}$.

<u>Proposition 6</u>. a) If X is a Hausdorff pointly-weakly-$\sigma\theta$-refinable space, then X is CC-closed;

b) if X is a Hausdorff countably compact space and $X \setminus x_0$ is weakly-$\sigma\theta$-refinable, then $t(x_0) \leq \aleph_0$.

<u>Proof</u>: a) immediately follows from Lemma 1.

To prove b) suppose $[A] \ni x_0$ and $B = \bigcup\{[S] : S \subseteq A\}$ then B is countably compact and $B \subseteq X \setminus x_0$. According to Lemma 1 $B = [B]$, hence $[A] \not\ni x_0$; a contradiction.

<u>Proposition 7</u>. Let $X = \bigcup_{i=1}^{\infty} X_i$ and for each i, X_i is a Hausdorff weakly-$\sigma\theta$-refinable and sequential space, then X is CC-closed.

<u>Proof</u>: Let A be a countably compact subspace of X and $A_i = A \cap X_i$, then A_i is closed in X_i, otherwise there exist $x_0 \in X_i \setminus A_i$ and a sequence $x_i \in A_i$ such that $x_i \to x_0$ but then $x_0 \in A$ and hence $x_0 \in A_i$; a contradiction. Therefore A_i is a weakly-$\sigma\theta$-refinable, and so A is also a weakly-$\sigma\theta$-refinable as a countable union of such spaces. Hence A is a bicompactum according to [8], therefore A is closed in X.

<u>Lemma 2</u>. Let X be a countably compact and CC-closed space, then

a) X is a space of countable tightness,

b) if $A \subseteq X$, then $|[A]| \leq |A|^{\aleph_0}$.

Proof: a) If $A \subseteq X$ and $B = \bigcup\{[S] : S \subseteq A \ |S| \leq \aleph_0\}$, then B is also countably compact and so $B = [B]$.

b) Let $A_0 = A$ and for every $\alpha < \beta < \omega_1$ we have already defined A_α. Let $A'_\beta = \bigcup\{A_\alpha : \alpha < \beta\}$ and $\mathcal{B} = \{S : S \subseteq A'_\beta$ and S be countable discrete in $A'_\beta\}$. Then $|\mathcal{B}| \leq |A|^{\aleph_0}$. For every $S \subseteq \mathcal{B}$ fix a point $x(S) \in [S] \setminus A'_\beta$ and put $A_\beta = A'_\beta \cup \{x(S) : S \in \mathcal{B}\}$. Then $[A] = \bigcup\{A_\beta : \beta < \omega_1\}$. Really, $\bigcup\{A_\beta : \beta < \omega_1\} \subset [A]$, and if $\bigcup\{A_\beta : \beta < \omega_1\}$ is not closed, then it is not countably compact, therefore there is a countable set S which is discrete in $\bigcup\{A_\beta : \beta < \omega_1\}$. But then there is $\beta_0 < \omega_1$ such that $S \subset A_{\beta_0}$ and so $x(S) \in [S]$ and $x(S) \in A_{\beta_0+1}$. This contradicts the fact that S is discrete in $\bigcup\{A_\beta : \beta < \omega_1\}$.

Proposition 8. Let X be a regular countably compact space with the property that each closed $F \subseteq X$ contains a point of countable character in F, then if X is CC-closed, then X is sequential.

Proof: Let $[A]_c$ be a sequential closure of A, and $[A]_c \neq [A]$. It follows that $[A]_c$ is not countably compact, so there is a countable $S \subset [A]_c$ which is discrete in $[A]_c$. Now the set $F = [S] \setminus S \subseteq [A] \setminus [A]_c$ and F is closed in X (because S is discrete in itself). Let x_0 be a point of countable character in F. Then x_0 is a point of countable character also in $[S]$, because $[S]$ is a regular and countably compact space, therefore there exists a sequence $\{x_n\} \subseteq S$ such that $x_n \to x_0$ and so $x_0 \in [A]_c$, a contradiction.

Proposition 9. ($2^{\aleph_1} > 2^{\aleph_0}$). Let X be a bicompactum. Then X is a CC-closed space iff X is a sequential space.

Let us prove a non-trivial part. Let X be a CC-closed space, then $t(X) \leq \aleph_0$ (according to Lemma 2 a)) and so $t(F) \leq \aleph_0$ for every closed $F \subseteq X$. Then according to a lemma of A.V. Arhangel'skii [6] there are countable $S \subseteq X$ and a G_δ in F bicompactum Φ such that $[S] \supseteq \Phi$. But according to Lemma 2 b) $|[S]| \leq 2^{\aleph_0}$, hence $|\Phi| \leq 2^{\aleph_0}$. Now if $2^{\aleph_1} > 2^{\aleph_0}$, then there is y_0 a G_δ-point in Φ and so it is a point of countable character in F. Now according to Proposition 8, X is sequential.

Corollary 4. ($2^{\aleph_1} > 2^{\aleph_0}$). If X is a bicompactum, $X = \bigcup_{i=1}^{\infty} X_i$ and for each i, X_i is a sequential weakly-$\delta\theta$-refinable space, then X is a sequential space.

It follows from Proposition 7 and Proposition 9.

Proposition 10. Let X be a pointly-$\delta\theta$-refinable bicompactum, then

a) $t(X) \leq \aleph_0$,

b) ($2^{\aleph_1} > 2^{\aleph_0}$) X is sequential.

It follows from Lemma 2 and Proposition 9.

Proposition 11 (main). Let X be a bicompactum and $X = \bigcup_{i=1}^{\infty} X_i$, then any of the following conditions implies that X is a sequential space with a dense set of G_δ-points;

a) for every i, X_i is a space with G_δ-diagonal,

b) for every i, X_i is a weakly-$\delta\theta$-refinable space with a countable pseudocharacter;

c) for every i, X_i is a pointly-metalindelöf space.

Proof: In any of these cases, each closed set $F \subseteq X$ has a G_δ-point (in F). Really, it follows from one theorem from [2] in the cases a) and b), while in the case c) for every $x_0 \in X$ we have $X \setminus \{x_0\} = \bigcup_{i=1}^{\infty} X_i \setminus \{x_0\}$, hence $X \setminus x_0$ is weakly-$\sigma\theta$-refinable, so according to Proposition 6 a) X is CC-closed and hence of countable tightness (Lemma 2 a)). Now, using Corollary 1 a) we receive the necessary fact.

Besides, in any of these cases X is a CC-closed space. Really, the case c) is clear. In the case b) it follows from the fact that $X \setminus \{x_0\} = \bigcup_{i=1}^{\infty} X_i \setminus \{x_0\}$ and so is a weakly-$\sigma\theta$-refinable space, as a countable union of such spaces and further from Proposition 6 a). In the case a) it follows from a theorem of Chaber [11]: if a regular countably compact space is the union of countably many spaces and each of them has a G_δ-diagonal, then X is a bicompactum.

Now $\bigcup X_i$ is a sequential space according to Proposition 8.

<u>Corollary 5</u>. Let X be a bicompactum, $X = \bigcup_{i=1}^{\infty} X_i$ and every X_i be embedded in some Σ_*-product of separable metric spaces, then X is a sequential bicompactum with a dense set of G_δ-points.

It follows from the fact that every Σ_*-product of separable metric spaces is hereditarily metalindelöf and from Proposition 11 c.

The last fact generalizes the well-known properties of Eberlein bicompacta. This result cannot be significantly improved, because such a bicompactum need not be a Fréchet-Uryson bicompactum. For example, the so-called separable Franklin bicompactum is such a space. On the other hand, there is a bicompactum which may be embedded even into the union of two Σ-products of $\mathcal{D}_{\infty} = \{0,1\}$, but does not have even a countable tightness. It is a space TW ($\omega_1 + 1$).

Problem: let X be a bicompactum and $X = X_1 \cup X_2$, where each X_i is embedded into some Σ_*-product of compacta. Does X be a Fréchet-Uryson bicompactum? Is X an Eberlein bicompactum? And if X_i are embedded into the same Σ_*-product?

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(Oblatum 28.6. 1979)