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MIXED VARIATIONAL FORMULATION OF UNILATERAL PROBLEMS

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Abstract: The mixed variational formulation of unilateral boundary value problems is derived and its finite element approximation is studied. Provided, the exact solution is smooth enough, the rate of convergence for both, the primal quantity as well as for associated Lagrange multiplier is obtained. Algorithm for effective computation is proposed.

Key words: Finite element method, mixed variational formulation.

Classification: 65N30

The application of finite element method for solving unilateral boundary value problems has been discussed by many authors (Glowinski-Lions-Tremilières), (Brezzi-Hager-Raviart), (Hlaváček), etc. Using the Ritz method, the original minimization problem over a convex set $K(\Omega)$ of functions, satisfying unilateral boundary conditions, is transformed into the minimization problem over the "finite element approximation" of $K(\Omega)$, where the conditions on $\partial\Omega$ are satisfied approximately only. Various methods of quadratic programming can be used for numerical solution. In this paper, another way is chosen. Proper dualization of constrains leads to the problem of finding the saddle-point of a certain Lagrangian. This formulation is used for the numerical solution of the original unilateral problem by finite element method. The error estimates for both components of the saddle-point are given. This appreach has two advantages:

- the Lagrange multiplier has obviously the physical meaning, so that its approximation is useful;
- using the Uzawa's algorithms, it is possible to carry out very economic calculations.

Let us mention the similar approach can be used for various unilateral problems, as the Signorini problem with or without friction, the contact problems of elastic bodies, etc., see (Haslinger, Hlaváček), (Nečas, Jarušek, Haslinger).

Functional context and notations. Let Ω be a domain with Lipschitz boundary $\partial\Omega$. We shall use the Sobolev space $H^k(\Omega)$ of functions, derivatives of which up to the order k exist and are square integrable in Ω. The usual norm of u in $H^k(\Omega)$ will be denoted by $\|u\|_k$. $H^{1/2}(\partial\Omega)$ denotes the space of traces of functions, belonging to $H^1(\Omega)$. It is the Hilbert space, equipped with the norm

$$\|\varphi\|_{1/2} = \inf_{\substack{v=\varphi \\ v\in H^1(\Omega)}} \|v\|_1 \quad .$$

We shall denote by $H^{-1/2}(\partial\Omega)$ the dual space to $H^{1/2}(\partial\Omega)$ and by $\| \ \|_{-1/2}$ the dual norm. The duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ will be denoted by $\langle \, , \, \rangle$.

1. Variational formulation of the problem. We consider the following model problem

(1.1) $$-\Delta u + u = f \quad \text{in } \Omega$$

with the unilateral boundary conditions

(1.2) $$u \geq g, \quad \partial u/\partial n \geq 0, \quad (u-g)\,\partial u/\partial n = 0 \quad \text{on } \partial\Omega ,$$

where $\partial u/\partial n$ denotes the normal derivative with respect to the outward normal n, $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$.

Let us introduce the convex set

(1.3) $$K(\Omega) = \{v \in H^1(\Omega) \mid v \geq g \quad \text{a.e. on } \partial\Omega\}$$

and the functional of potential energy

(1.4) $$J(v) = \|v\|_1^2 - 2(f,v)_o,$$

where $(\, , \,)_o$ denotes the scalar product in $L^2(\Omega)$.

Then the problem

($\mathcal{P}$) $$\begin{cases} \text{find } u \in K(\Omega) \text{ such that} \\ J(u) \leq J(v) \quad \forall v \in K(\Omega) \end{cases}$$

represents the variational formulation of (1.1) and (1.2).

<u>Theorem 1.1</u>. There exists a unique solution of ($\mathcal{P}$) and u can be characterized through the relations:

(1.5) $$\begin{cases} u \in K(\Omega), \\ (u,v-u)_1 \equiv (\operatorname{grad} u, \operatorname{grad} (v-u))_o + (u,v-u)_o \geq (f,v-u)_o \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in K(\Omega). \end{cases}$$

<u>Proof</u>: see [2].

<u>Mixed formulation of</u> ($\mathcal{P}$). Let

(1.6) $$\Lambda(\partial\Omega) = \{\mu \in H^{-1/2}(\partial\Omega) \mid \langle \mu, v\rangle \geq 0 \quad \forall v \in H^{1/2}(\partial\Omega), \; v \geq 0\}.$$

It is easy to see that

(1.7) $$v \in K(\Omega) \Longleftrightarrow v \in H^1(\Omega) \text{ and } \langle \mu, v-g\rangle \geq 0 \quad \forall \mu \in \Lambda(\partial\Omega).$$

Let us define the following problem, the so called mixed variational formulation of $(\mathcal{P})$:

$$(\mathcal{P}^*)\begin{cases}\text{find } (u^*,\lambda^*)\in H^1(\Omega)\times\Lambda(\partial\Omega), \text{ satisfying}\\ (u^*,v)_1-\langle\lambda^*,v\rangle=(f,v)_0 \qquad \forall v\in H^1(\Omega)\\ \langle\lambda-\lambda^*,u^*\rangle\geq\langle\lambda-\lambda^*,g\rangle \qquad \forall\lambda\in\Lambda(\partial\Omega).\end{cases}$$

Theorem 1.2. There exists a unique solution of $(\mathcal{P}^*)$ and

$$(u^*,\lambda^*)=(u,\partial u/\partial n),$$

where u is the solution of $(\mathcal{P})$. (For the proof see [2].)

Remark 1.1. $(u,\partial u/\partial n)$ can be characterized equivalently as the saddle-point of the Lagrangian $\mathcal{L}(v,\lambda)$ over $H^1(\Omega)\times\Lambda(\partial\Omega)$, where

(1.8) $$\mathcal{L}(v,\lambda)=J(v)-\langle\lambda,v-g\rangle.$$

2. Finite element approximation of $(\mathcal{P}^*)$. To propose a consistent mixed finite element analysis, we shall consider straight triangular elements only and therefore study problems on polygonal domains. For simplicity we restrict ourselves to plane polygonal domains.

Thus let $\Omega\subset R^2$ be a polygonal bounded domain. We consider a regular family of triangulations $\{\mathcal{T}_h\}$ of Ω, where $h=\max\operatorname{diam}T$, $T\in\mathcal{T}_h$. Let $\{\mathcal{T}_H\}$ be a family of partitions of $\partial\Omega$, independent (for the moment) on $\mathcal{T}_h$, i.e.

$$\partial\Omega=\bigcup_{i=1}^{m}a_ia_{i+1},\quad a_1=a_{m+1}.$$

We denote by H_i = length a_ia_{i+1} and $H=\max H_i$. Next we shall consider a regular family $\{\mathcal{T}_H\}$ in the following sense:

there exists a constant $\alpha_0 > 0$ such that $H/H_i \leq \alpha_0$, $i=1,\dots,m$.

Let us define the spaces

$$V_h(\Omega) = \{ v \in C(\bar{\Omega}) | \ v|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h \} ,$$

$$\tilde{\Lambda}_H(\partial\Omega) = \{ \mu_H \in L^2(\partial\Omega) | \ \mu_H|_{a_i a_{i+1}} \in P_0(a_i a_{i+1}), \\ i=1,\dots,m \},$$

where $P_1(T)$, $P_0(a_i a_{i+1})$ denotes the space of all polynomials of degree 1 in 2 variables and the constant functions on $a_i a_{i+1}$, respectively.

Let us define the following problem

$$(\mathcal{P}_h^*) \begin{cases} \text{find } (u_h^*, \lambda_H^*) \in V_h(\Omega) \times \Lambda_H(\partial\Omega) \text{ such that} \\ (u_h^*, v_h)_1 - \langle \lambda^*, v_h \rangle = (f, v_h)_0 \qquad \forall v_h \in V_h(\Omega) \\ \langle \mu_H - \lambda_H^*, u_h^* \rangle \geq \langle \mu_H - \lambda_H^*, g \rangle \quad \forall \mu_H \in \Lambda_H(\partial\Omega), \end{cases}$$

with

$$\langle \mu_H, v_R \rangle = \int_{\partial\Omega} \mu_H v_h \, ds \qquad \forall v_h \in V_h(\Omega), \ \mu_H \in \Lambda_H(\partial\Omega),$$

$$\Lambda_H(\partial\Omega) = \{ \mu_H \in \tilde{\Lambda}_H(\partial\Omega) \ | \ \mu_H \geq 0 \quad \text{on } \partial\Omega \} .$$

$(\mathcal{P}_h^*)$ is the approximation of $(\mathcal{P}^*)$.

3. Existence and uniqueness of the solution of $(\mathcal{P}_h^*)$.

In what follows, we shall study the existence and the uniqueness of the solution of $(\mathcal{P}_h^*)$. Let us set

$$(3.1) \qquad \mathcal{K}(h,H) = \{ v_h \in V_h(\Omega) \ | \ \langle \mu_H, v_h \rangle \geq \langle \mu_H, g \rangle \\ \forall \mu_H \in \Lambda_H(\partial\Omega) \}.$$

Then $\mathcal{K}(h,H)$ is the finite-dimensional approximation of $K(\Omega)$. Next, let the pair $(u_h^*, \lambda_H^*) \in V_h(\Omega) \times \Lambda_H(\partial\Omega)$ be the solution of $(\mathcal{P}^*_h)$. Then u^*_h is the unique solution of the problem:

$$\text{(3.2)}\qquad \begin{cases} \text{find } u^*_h \in \mathcal{K}(h,H) \text{ such that} \\ (u^*_h, v_h - u^*_h)_1 \geq (f, v_h - u^*_h)_o \qquad \forall\, v_h \in \mathcal{K}(h,H). \end{cases}$$

On the other hand, a pair $(u^*_h, \lambda^*_H) \in V_h(\Omega) \times \Lambda_H(\partial\Omega)$ can be characterized as the saddle-point of the Lagrangian $\mathcal{L}$ over the set $V_h(\Omega) \times \Lambda_H(\partial\Omega)$, where $\mathcal{L}$ is defined by

$$\text{(3.3)}\qquad \mathcal{L}(v_h, \mu_H) = J(v_h) - \langle \mu_H, v_h - g \rangle .$$

It means, the first component of the saddle-point (u^*_h, λ^*_H) is uniquely determined. In order to prove the uniqueness of λ^*_H we need the following result.

<u>Lemma 3.1</u>. Let h/H be sufficiently small. Then

$$\text{(3.4)}\qquad \begin{cases} \sup\limits_{V_h(\Omega)} \dfrac{\langle \mu v_h \rangle}{\|v_h\|_1} \geq \beta \|\mu\|_{-1/2} \\ \sup\limits_{V_h(\Omega)} \dfrac{\langle \mu v_h \rangle}{\|v_h\|_1} \geq \beta H^{1/2} \|\mu\|_{L^2(\partial\Omega)} \end{cases}$$

hold for any $\mu \in \Lambda_H(\partial\Omega)$ with a positive constant $\beta > 0$.

<u>Proof</u>. As

$$\|\mu\|_{-1/2} = \sup_{H^1(\Omega)} \frac{\langle \mu, v \rangle}{\|v\|_1} ,$$

we obtain (see [8])

$$\text{(3.5)}\qquad \|\mu\|_{-1/2} = \frac{\langle \mu, \bar{v} \rangle}{\|\bar{v}\|_1} = \|\bar{v}\|_1 ,$$

where $\bar{v} \in H^1(\Omega)$ is the unique solution of the boundary value problem:

$$-\Delta v + v = 0 \quad \text{in } \Omega$$
$$\partial v / \partial n = \mu \quad \text{on } \partial\Omega .$$

Let $\mu \in \tilde{\Lambda}_H(\partial\Omega)$. Then $\mu \in H^{-1/2+\varepsilon}(\partial\Omega)$ for $\forall \varepsilon \in \langle 0,1)$ and the following regularity holds

$$(3.6) \qquad \|\bar{v}\|_{1+\varepsilon} \leq c(\varepsilon) \|\mu\|_{-1/2+\varepsilon} .$$

Furthermore we write

$$(3.7) \qquad \sup_{V_h(\Omega)} \frac{\langle \mu, v_h \rangle}{\|v_h\|_1} \geq \frac{\langle \mu, \bar{v}_h \rangle}{\|\bar{v}_h\|_1} = \|\bar{v}_h\|_1$$

where $\bar{v}_h$ is the Galerkin approximation of $\bar{v}$ on $V_h(\Omega)$. Applying the triangle inequality we obtain

$$\|\bar{v}_h\|_1 = \|\bar{v}_h - \bar{v} + \bar{v}\|_1 \geq \|\bar{v}\|_1 - \|\bar{v}_h - \bar{v}\|_1 .$$

Together with (3.7) we obtain

$$(3.8) \qquad \|\bar{v}\|_1 \leq \sup_{V_h(\Omega)} \frac{\langle \mu, v_h \rangle}{\|v_h\|_1} + \|\bar{v}_h - \bar{v}\|_1 .$$

On the other hand, using the error estimates of classical finite elements, (3.6) as well as the inverse hypothesis between $H^{-1/2}(\partial\Omega)$ and $H^{-1/2+\varepsilon}(\partial\Omega)$ for $\mu \in \tilde{\Lambda}_H(\partial\Omega)$, we obtain

$$\|\bar{v}_h - \bar{v}\|_1 \leq c\, h^{\varepsilon} \|\bar{v}\|_{1+\varepsilon} \leq c(\varepsilon)\, h^{\varepsilon} \|\mu\|_{-1/2+\varepsilon} \leq c(h/H)^{\varepsilon} \|\mu\|_{-1/2} .$$

From this, (3.8) and (3.5), the inequality (3.4) holds. In order to prove the second inequality in (3.4), we take into account the inverse hypothesis between $L^2(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ for $\mu \in \tilde{\Lambda}_H(\partial\Omega)$. We can write

$$\|\mu\|_0 \leq c\, H^{-1/2} \|\mu\|_{-1/2} \qquad \forall \mu \in \tilde{\Lambda}_H(\partial\Omega).$$

Hence we obtain the second inequality in (3.4).

<u>Theorem 3.1</u>. There exists a unique solution $(u_h^*, \lambda_H^*) \in V_h(\Omega) \times \Lambda_H(\partial\Omega)$ of $(\mathcal{P}_h^*)$.

<u>Proof</u>. We can apply the abstract result from [1].

4. <u>Error estimates</u>. Now, we shall study the rate of convergence of u_h^* to u^* and λ_H^* to λ^*, provided the exact solution u of $(\mathcal{P})$ is smooth enough. First, we derive the abstract error estimates. To this end, we give another, equivalent form of $(\mathcal{P}^*)$. Let

$$\mathcal{H}(\Omega) = H^1(\Omega) \times H^{-1/2}(\partial\Omega)$$

be the Hilbert space with the norm

(4.1) $\quad |||V||| = (\|v\|_1^2 + \|\mu\|_{-1/2}^2)^{1/2}, \quad V = (v, \mu) \in \mathcal{H}(\Omega).$

Let

(4.2) $\quad \mathcal{A}(U,V) = (u,v)_1 - \langle \lambda, v \rangle + \langle \mu, u \rangle , \quad U = (u, \lambda)$

be a continuous, bilinear form on $\mathcal{H}(\Omega) \times \mathcal{H}(\Omega)$ and $\mathbb{K}(\Omega) = H^1(\Omega) \times \Lambda(\Omega)$ be a non-empty closed convex subset of $\mathcal{H}(\Omega)$. We denote by M a positive constant such that

$$|\mathcal{A}(U,V)| \leq M \, |||U||| \; |||V||| \qquad \forall U,V \in \mathcal{H}(\Omega).$$

On the other hand we see that

(4.3) $\qquad \mathcal{A}(U,U) = (u,u)_1 \qquad \forall U \in \mathcal{H}(\Omega).$

Now we consider the following variational problem

$$(\tilde{\mathcal{P}}) \qquad \begin{cases} \text{find } U^* = (u^*, \lambda^*) \in \mathbb{K}(\Omega) \text{ such that} \\ \mathcal{A}(U^*, V - U^*) \geq \mathcal{F}(V - U^*) \qquad \forall V \in \mathbb{K}(\Omega), \end{cases}$$

where $\mathcal{F}(V) = (f,v)_0 + \langle \mu, g \rangle$.

Immediately we obtain

<u>Lemma 4.1</u>. The problem $(\tilde{\mathcal{P}})$ is equivalent to $(\mathcal{P}^*)$.

Let us set

$$K(h,H) = V_h(\Omega) \times \Lambda_H(\partial\Omega).$$

Then $K(h,H)$ is the internal approximation of $\mathbb{K}(\Omega)$. The approximation of $(\mathcal{P})$ is the following:

$$(4.4)\qquad \begin{cases} \text{find } U_h^* = (u_h^*, \lambda_H^*) \in K(h,H) \text{ such that} \\ \mathcal{A}(U_h^*, V_h - U_h^*) \geq \mathcal{F}(V_h - U_h^*) \quad \forall V_h \in K(h,H). \end{cases}$$

Again, it is easy to see that (4.4) is equivalent to $(\mathcal{P}_h^*)$.

Lemma 4.2. It holds:

$$(4.5)\qquad \| u^* - u_h^* \|_1^2 = (u^* - u_h^*, u^* - u_h^*)_1 \leq \{ \mathcal{A}(U^* - U_h^*, U^* - V_h) + $$
$$+ \mathcal{A}(U^*, V_h - U^*) + \mathcal{F}(U^* - V_h)\} \quad \forall V_h \in K(h,H).$$

Proof. We deduce from the relation (4.3)

$$(u^* - u_h^*, u^* - u_h^*)_1 = \mathcal{A}(U^* - U_h^*, U^* - U_h^*) =$$
$$= \mathcal{A}(U^* - U_h^*, U^* - V_h) + \mathcal{A}(U^*, V_h - U^*) + \mathcal{A}(U^*, U^* - U_h^*) +$$
$$+ \mathcal{A}(U_h^*, U_h^* - V_h) \leq \mathcal{A}(U^* - U_h^*, U^* - V_h) + \mathcal{A}(U^*, V_h - U^*) +$$
$$+ \mathcal{F}(U_h^* - V_h) + \mathcal{F}(U^* - U_h^*).$$

Lemma 4.3. It holds:

$$(4.6)\qquad \| \lambda^* - \lambda_H^* \|_{-1/2} \leq c \{ \| u^* - u_h^* \|_1 + \inf_{\Lambda_H} \| \lambda^* - \mu_H \|_{-1/2} \}$$

$$(4.7)\qquad \| \lambda^* - \lambda_H^* \|_0 \leq c \{ \| \lambda^* - \mu_H \|_0 + H^{-1/2} \| \lambda^* - \mu_H \|_{-1/2} +$$
$$+ H^{-1/2} \| \lambda^* - \lambda_H^* \|_{-1/2} \} \text{ if } \lambda^* \in L^2(\partial\Omega).$$

Proof. Using the definition of $(\mathcal{P}_h^*)$ we obtain

$$\langle \mu_H - \lambda_H^*, v_h \rangle = \langle \mu_H, v_h \rangle - \langle \lambda_H^*, v_h \rangle = \langle \mu_H, v_h \rangle +$$
$$+ (f, v_h)_0 - (u_h^*, v_h)_1 = \langle \mu_H, v_h \rangle + (u^* - u_h^*, v_h)_1 - \langle \lambda^*, v_h \rangle \leq$$
$$\leq c \{ \| u^* - u_h^* \|_1 + \| \mu_H - \lambda^* \|_{-1/2} \} \| v_h \|_1.$$

Now we can write

$$\sup_{V_h(\Omega)} \frac{\langle \mu_H - \lambda^*_H, v_h\rangle}{\|v_h\|_1} \le \|u^* - u_h^*\|_1 +$$

$$+ \|\mu_H - \lambda^*\|_{-1/2} \qquad \forall \mu_H \in \Lambda_H(\partial\Omega).$$

From this and (3.4) it follows

$$\|\mu_H - \lambda^*_H\|_{-1/2} \le c\,\{\|u^* - u_h^*\|_1 + \|\lambda^* - \mu_H\|_{-1/2}\} \qquad \forall \mu_H \in \Lambda_H(\partial\Omega) \tag{4.8}$$

$$\|\mu_H - \lambda^*_H\|_0 \le c\, H^{-1/2}\ \|\mu_H - \lambda^*_H\|_{-1/2}\ . \tag{4.9}$$

Now (4.8),(4.9) together with triangle inequalities

$$\|\lambda^* - \lambda^*_H\|_{-1/2} \le \|\lambda^* - \mu_H\|_{-1/2} + \|\mu_H - \lambda^*_H\|_{-1/2}$$

$$\|\lambda^* - \lambda^*_H\|_0 \le \|\lambda^* - \mu_H\|_0 + \|\mu_H - \lambda^*_H\|_0 \text{ if } \lambda^* \in L^2(\partial\Omega)$$

result in (4.6) and (4.7).

<u>Lemma 4.4</u>. It holds

$$\|u^* - u_h^*\|_1^2 \le c\{\inf_{V_h(\Omega)} \|u^* - v_h\|_1^2 + \inf_{\Lambda_H} \{\|\lambda^* - \mu_H\|_{-1/2}^2 + \langle \mu_H - \lambda^*, u^* - g\rangle\}\}. \tag{4.10}$$

<u>Proof</u>. From (4.5) we deduce

$$\|u^* - u_h^*\|_1^2 \le M\{\varepsilon\,|||U^* - U_h^*|||^2 + \frac{1}{\varepsilon}\,|||U^* - V_h|||^2\} +$$
$$+ (u^*, v_h - u^*)_1 - \langle \lambda^*, v_h - u^*\rangle + \langle \mu_H - \lambda^*, u^*\rangle +$$
$$+ (f, u^* - v_h)_0 + \langle \lambda^* - \mu_H, g\rangle \le M\{\varepsilon\|u^* - u_h^*\|_1^2 +$$
$$+ \varepsilon\|\lambda^* - \lambda^*_H\|_{-1/2}^2 + \frac{1}{\varepsilon}\|u^* - v_h\|_1^2 + \frac{1}{\varepsilon}\|\lambda^* - \mu_H\|_{-1/2}^2 +$$
$$+ (-\Delta u^* + u^*, v_h - u^*)_0 + \langle \partial u/\partial n, v_h - u^*\rangle - \langle \lambda^*, v_h - u^*\rangle +$$
$$+ \langle \mu_H - \lambda^*, u^* - g\rangle + (f, u^* - v_h)_0 \le M\{\varepsilon\|u^* - u_h^*\|_1^2 +$$
$$+ \frac{1}{\varepsilon}\|u^* - v_h\|_1^2\} + \varepsilon\|\lambda^* - \lambda^*_H\|_{-1/2}^2 +$$

$+\frac{1}{\varepsilon}\ \|\lambda^* - \mu_H\|^2_{-1/2} + \langle \mu_H - \lambda^*, u^* - g\rangle\}.$

Taking $\varepsilon > 0$ sufficiently small, we obtain (4.11)

<u>Theorem 4.1</u>. Let u^* denote the solution of $(\mathcal{P})$ such that

(j) $u^* \in K(\Omega) \cap H^2(\Omega)$;

(jj) $u^*, g \in H^1_\infty(a_i a_{i+1})$, $i=1,\dots,m$,

(jjj) the set of points from $\partial\Omega$, where u^* changes from $u^* > g$ to $u^* = g$ is finite.

Then

(4.11) $\quad \|u^* - u_h^*\|_1 \leq c(u^*, g, \lambda^*)(h + H)$

(4.12) $\quad \|\lambda^* - \lambda^*_H\|_{-1/2} \leq c(u^*, g, \lambda^*)\ (h + H)$

(4.13) $\quad \|\lambda^* - \lambda^*_H\|_{-1/2} \leq c(u^*, g, \lambda^*)\ H^{-1/2}(h + H).$

<u>Proof</u>. We set $v_h = r_h u^* \in V_h(\Omega)$, where $r_h u^*$ denotes the piecewise linear Lagrange interpolate of u^*. From (j) we deduce ([3])

(4.14) $$\|u^* - r_h u^*\|_1 = O(h).$$

As $u^* \in H^2(\Omega)$, $\partial u^*/\partial n = \lambda^* \in H^{1/2}(a_i a_{i+1})$, $i=1,\dots,m$. Let $\bar{\mu}_H \in \mathcal{X}_H(\partial\Omega)$ be the orthogonal L^2-projection of λ^* on $\mathcal{X}_H(\partial\Omega)$. It is readily seen that $\bar{\mu}_H \in \Lambda_H(\partial\Omega)$ and in addition

(4.15) $$\|\lambda^* - \bar{\mu}_H\|_{-1/2} = O(H).$$

Finally it remains to estimate the term $\langle \bar{\mu}_H - \lambda^*, u^* - g\rangle$. To this end let

$$\partial\Omega_0 = \{x \in \partial\Omega \mid u^*(x) = g(x)\}$$

$$\partial\Omega_t = \{x \in \partial\Omega \mid u^*(x) > g(x)\} .$$

Now, if $a_i a_{i+1} \subset \partial\Omega_o$, we have

$$(4.16) \qquad \int_{a_i a_{i+1}} (\bar{\mu}_H - \lambda^*)(u^* - g)\, ds = 0 .$$

On the other hand, if $a_i a_{i+1} \subset \partial\Omega_t$, we have $\lambda^* = 0$, consequently $\bar{\mu}_H$ being the orthogonal projection is equal to zero on $a_i a_{i+1}$. Therefore (4.16) holds again.

Finally let the interior of $a_i a_{i+1}$ contain both the points, from $\partial\Omega_o$ and $\partial\Omega_t$. Then

$$\Big| \int_{a_i a_{i+1}} (u^* - g)(\lambda^* - \bar{\mu}_H)\, ds \Big| \le \| u^* - g\|_{L_\infty(a_i a_{i+1})}$$

$$\int_{a_i a_{i+1}} |\lambda^* - \bar{\mu}_H|\, ds \le \|u^* - g\|_{L_\infty(a_i a_{i+1})} H^{1/2} \|\lambda^* -$$

$$- \bar{\mu}_H\|_{L^2(a_i a_{i+1})} .$$

As $\lambda^* \in H^{1/2}(a_i a_{i+1})$, we have

$$\|\lambda^* - \bar{\mu}_H\|_{L^2(a_i a_{i+1})} = O(H^{1/2}).$$

By virtue of (jj) we get

$$u^* - g \in H^1_\infty(a_i a_{i+1}).$$

Since $u^* - g = 0$ in some point of $a_i a_{i+1}$, we have

$$\|u^* - g\|_{L_\infty(a_i a_{i+1})} = O(H).$$

Combining the two previous estimations, we obtain

$$\Big| \int_{a_i a_{i+1}} (u^* - g)(\bar{\mu}_H - \lambda^*)\, ds \Big| = O(H^2).$$

Now, using the fact that the number of segments, containing both the points of $\partial\Omega_o$, $\partial\Omega_t$, is bounded above independently of H. Hence

$$\langle \bar{\mu}_H - \lambda^* , u^* - g \rangle = O(H^2).$$

From this, (4.14) and (4.15), the estimation (4.11) follows. (4.12) and (4.13) are consequences of (4.11),(4.6) and (4.7).

<u>Remark 4.1</u>. If the regularity assumptions on u are weaker, the rate of convergence is lower, of course. If there are no regularity assumptions on u, one can prove the convergence of (u_h^*, λ^*_H) to (u^*, λ^*) in $\mathcal{H}(\Omega)$ only, without the information on its rate.

5. <u>Numerical algorithm for solving</u> $(\mathcal{P}^*_h)$. Our aim is to find the saddle-point of the Lagrangian

$$\mathcal{L}(v_h, \mu_H) = 1/2(v_h, v_h)_1 - (f, v_h)_o - \langle \mu_H, v_h - g \rangle$$

on the set $V_h(\Omega) \times \Lambda_H(\partial\Omega)$. To this end we use the Uzawa's algorithm.

This algorithm appears as the standard gradient method of the optimization theory, applied to the dual problem. It is based on the construction of two sequences of elements

$$u^{(n)} \in V_h(\Omega), \quad \mu^{(n)} \in \Lambda_H(\partial\Omega)$$

(indices are omitted), defined in the following way:

we start with any $\mu^{(o)} \in \Lambda_H(\partial\Omega)$ (given arbitrarily), we calculate $u^{(o)}$, then $\mu^{(1)}, u^{(1)}$, etc.

$\mu^{(n)}$ being known, $u^{(n+1)}$ is defined as the element of $V_h(\Omega)$ satisfying

$$(5.1) \qquad (u^{(n+1)}, v)_1 = (f, v)_o + \langle \mu^{(n)}, v \rangle \qquad \forall v \in V_h(\Omega).$$

Then we define

$$\mu^{(n+1)} = P_{\Lambda_H}(\mu^{(n)} - \rho_n(u^{(n)} - g)), \tag{5.2}$$

where $\rho_n > 0$ is some positive constant and P_{Λ_H} is the projection of $\tilde{\Lambda}(\partial\Omega)$ on $\Lambda_H(\partial\Omega)$. It is easy to see that

$$P_{\Lambda_H}(f)|_{a_i a_{i+1}} = \max(0, \pi_i(f)), \tag{5.3}$$

where $\pi_i(f)$ denotes the mean value of f on $a_i a_{i+1}$.

The algorithm (5.1) is very effective from the practical point of view. The components of the vector on the right hand side of (5.1), corresponding to nodal points, lying on $\partial\Omega$, change only. This fact can be used for economic computation. Let the numbering of nodes of $\mathcal{T}_h$ be such that the vector α of unknowns can be arranged as follows:

$$\alpha = \begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix}$$

where α' denotes the unknowns, associated to nodes from Ω, α' from the boundary $\partial\Omega$, respectively.
Then (5.1) can be written in the matrix form

$$[A]\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \mathcal{F}' \\ \mathcal{F}'' \end{pmatrix}$$

where $[A]$ is the stiffness matrix and $\mathcal{F}'$, $\mathcal{F}''$ are defined in a similar way as (α', α''). Using the partial elimination, we obtain

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} g' \\ g'' \end{pmatrix}$$

with the above decomposition of the vector α, Uzawa's algorithm can be applied to much smaller system of linear equations

$$A_3\alpha'' = g'' ,$$

where the number of unknowns is equal to a number of nodes of $\mathcal{T}_h$ on $\partial\Omega$. This decomposition of the vector α is not necessary, if we use a modification of the frontal solution method for solving linear equations.

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