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APPROXIMATELY NORMAL FUNCION ALGEBRAS WHICH ARE LOCAL

Wesley KOTZÉ

Abstract: A function algebra is called local if continuous functions which belong locally to the function algebra actually belong to the algebra. It has long been known that all normal function algebras are local. The corresponding question is not fully answered for approximately normal function algebras. We show here that not only are all approximately normal analytic function algebras local, but also approximately normal function algebras which are integral domains.

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1. Definitions. A function algebra A is a subalgebra of C(X) [the algebra of all complex-valued continuous functions on a compact Hausdorff space X], which is uniformly closed, contains the constant functions, and separates points.

A function algebra A is normal on X if for every pair of disjoint closed subsets F_1 and F_2 on X an $f \in A$ exists such that $f = 0$ on F_1 and $f = 1$ on F_2.

A function algebra A is approximately normal on X if for every pair of disjoint closed subsets F_1 and F_2 of X, and for every $\varepsilon > 0$, there exists an $f \in A$ such that

$|f(x)| < \varepsilon$ on F_1 and $|1 - f(x)| < \varepsilon$ on F_2.

A function algebra A is analytic on X if every function in A which vanishes on a non-empty open subset of X, vanishes identically.

A function algebra A is an integral domain if $f \in A$, $g \in A$, $f \not\equiv 0$, $g \not\equiv 0$ imply that $fg \not\equiv 0$.

A function algebra A is anti-symmetric on X if $f \in A$ and f real-valued on X imply that f is a constant on X.

It can easily be shown that for a function algebra A on X, the following implications hold:

A is analytic $\Longrightarrow$ A is an integral domain $\Longrightarrow$ A is anti-symmetric.

A function $f \in C(X)$ belongs locally to a function algebra A if for every $x \in X$, there exists a neighbourhood U_x of x and $g_x \in A$ such that $f(x) = g_x(x)$ on U_x; or by virtue of the compactness of X, f belongs locally to A if and only if there exists a finite open cover $U_1, U_2, \ldots, U_n$ of X and $g_1, g_2, \ldots, g_n \in A$ such that $f(x) = g_i(x)$ on U_i, $i=1\ldots n$ (i.e. $f|_{U_i} \in A|_{U_i}$).

A function algebra is local if it contains every function belonging locally to A.

The notion of an $f \in C(X)$ which is locally approximable by a function algebra A is defined in the analogous way: if for every $x \in X$ there exists a neighbourhood U_x of x such that $f|_{U_x} \in (A|_{U_x})^{-}$, the closure of the restriction of A to U_x.

M. Krein proved [1] that all normal function algebras are local, and D. Wilken [3] extended this result by showing that a function locally approximable by a normal function algebra A is in A. It is also known through counterexamples (see e.g. [3]) that a function locally approximable by an approximately normal function algebra A is not necessarily in A.

The question as to whether approximately normal function algebras are local has so far only been partially answered (see e.g. [2]). In this paper we pursue this question.

2. The following result by Wilken [3] is fundamental to our subsequent arguments:

2.1. Lemma. *Let A be an approximately normal function algebra on X. Let $f \in C(X)$ and $X = U_1 \cup U_2$, U_1 and U_2 open and $f(x) = g_i(x)$ on U_i $(i=1,2)$, $g_i \in A$; then $f \in A$.*

We conclude that approximately normal analytic, integral domain and anti-symmetric function algebras can only occur on connected spaces:

2.2. Proposition. *If A is an approximately normal anti-symmetric function algebra on X, then X is connected.*

Proof. Suppose X is disconnected, i.e. there exists a proper subset S of X which is both open and closed.

The function $h = \begin{cases} 0 \text{ on } S \\ 1 \text{ on } X \setminus S \end{cases}$

is continuous on X and by Lemma 2.1 belongs to A. This is however not possible since the only real-valued functions in an anti-symmetric function algebra are the constants. Thus X must be connected.

2.3. Theorem. An analytic function algebra A on a connected space X is local.

Proof. If $f \in C(X)$ belongs locally to the function algebra A, we have $U_1,\dots,U_n$ an open cover of X and $g_1,\dots$ $\dots,g_n \in A$ such that $f(x) = g_i(x)$ on U_i. Since X is connected every U_i intersects at least one other U_j; moreover we have, after possible re-labelling that $U_1 \cap U_2 \neq \emptyset$ and generally $(U_1 \cup \dots \cup U_i) \cap U_{i+1} \neq \emptyset$ for $i=1,\dots,n-1$. Since A is analytic this gives that $g_1 \equiv g_2 \equiv \dots \equiv g_n$. Thus $f(x) = g_1(x)$ on X and so $f \in A$.

2.4. Corollary. An approximately normal analytic function algebra is local.

This follows immediately from Proposition 2.2 and Theorem 2.3.

A generalization of Corollary 2.4 is given by the following Theorem - a result which Wells proved [2] for the case when X is the unit circle in the complex plane.

2.5. Theorem. An approximately normal function algebra which is an integral domain is local.

Proof. Suppose that $f \in C(X)$ belongs locally to A but is not in A. So we have an open cover $U_1,\dots,U_n$ of X and associated $g_1,\dots,g_n \in A$ with $f(x) = g_i(x)$ on U_i. Since, by Proposition 2.2, X is connected, every U_i intersects at least one other U_j. Put $W_i = \{x \in X; f(x) = g_i(x)\}$. Then $U_i \subset W_i^o$ and $\{W_i;\ i=1,\dots,n\}$ will cover X. (In fact $\{W_i^o;\ i=1,\dots,n\}$ will cover X.) Let $V_1,\dots,V_m$ be the minimal subcover of the cover $\{W_i;\ i=1,\dots,n\}$; also minimal in the sense that if $W_i \cap W_j \neq \emptyset$ and $g_i = g_j$ on $W_i \cap W_j$, then $W_i \cup W_j$

is to be labelled a V_i. Let $\tilde{g}_1,\ldots,\tilde{g}_m$ be the elements of A associated with the $V_1,\ldots,V_m$ so that $f(x) = \tilde{g}_i(x)$ on V_i. In view of the assumption that $f \notin A$, $m \geq 2$. (If we also take Lemma 2.1 into consideration we conclude that $m \geq 3$.) Then consider the function

$$f_1 = \begin{cases} 0 & \text{on } V_2 \cup \ldots \cup V_m \\ (\tilde{g}_1 - \tilde{g}_{11})(\tilde{g}_1 - \tilde{g}_{12})\ldots(\tilde{g}_1 - \tilde{g}_{1k_1}) & \text{on } V_1 \end{cases}$$

$$= \begin{cases} 0 & \text{on } X \setminus \bar{O}_1 \\ (\tilde{g}_1 - \tilde{g}_{11})(\tilde{g}_1 - \tilde{g}_{12})\ldots(\tilde{g}_1 - \tilde{g}_{1k_1}) & \text{on } V_1^o \end{cases}$$

where $O_1 = V_1 \setminus (V_{11} \cup \ldots \cup V_{1k_1}) = V_1 \setminus (V_2 \cup \ldots \cup V_m)$
$= X \setminus (V_2 \cup \ldots \cup V_m)$.

O_1 is open and non-empty by virtue of the minimal property of the cover $V_1,\ldots,V_m$. Furthermore $O_1 \subset V_1^o$ and

$$\begin{aligned} X \setminus \bar{O}_1 &= (X \setminus O_1)^o \\ &= (V_2 \cup V_3 \cup \ldots \cup V_m)^o \\ &\supset (V_2^o \cup V_3^o \cup \ldots \cup V_m^o \end{aligned}$$

hence $(X \setminus \bar{O}_1) \cup V_1^o = X$.

So $f_1 \in C(X)$ and is not identically zero by the construction of the V_i. By Lemma 2.1, $f_1 \in A$.

Define similarly

$$f_2 = \begin{cases} 0 & \text{on } (V_1 \cup \ldots \cup V_m) \setminus V_{11} \\ (\tilde{g}_{11} - \tilde{g}_{111})(\tilde{g}_{11} - \tilde{g}_{112})\ldots(\tilde{g}_{11} - \tilde{g}_{11k_{11}}) & \text{on } V_{11} \end{cases}$$

where the $V_{111}, V_{112},\ldots,V_{11k_{11}}$ which are associated with the $\tilde{g}_{111},\tilde{g}_{112},\ldots,\tilde{g}_{11k_{11}}$ are those V_i (other than V_{11}) which intersect V_{11}. (One of them will of course be V_1.) As in the case of f_1, $f_2 \in A$ and is not identically zero. However $f_1 f_2 = 0$. This contradicts the fact that A is an integral domain.

The question remains as to whether Theorem 2.5 can be extended to anti-symmetric function algebras. Even if it could only be proved that an anti-symmetric function algebra with maximal ideal space the unit circle (such an algebra is approximately normal on the unit circle) is local, we could conclude from a result by Wells ("A function algebra with maximal ideal space the unit circle is either anti-symmetric or local", see [2]) that any function algebra with maximal ideal space the unit circle is local - a statement which is true if the maximal ideal space is the unit interval. This will be in agreement with the old conjecture (originated by Gelfand in 1957) that the only function algebra with maximal ideal space the unit circle or interval is the uniform algebra of all continuous functions on the circle (respectively interval) itself.

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