Jindřich Nečas; Oldřich John; Jana Stará

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COUNTEREXAMPLE TO THE REGULARITY OF WEAK SOLUTION OF ELLIPTIC SYSTEMS

J. NEČAS, O. JOHN, J. STARÁ

Abstract: In the paper there will be given an example of nonlinear elliptic system

(1) $\sum_{i=1}^{n} D^i(a_r^i(\text{grad } u)) = 0,\ r = 1,\dots,m$ on $\Omega = \{x \in R_n,\ |x|<1\}$, $u = u_0$ on $\partial\Omega$,

having analytic coefficients and unique solution with discontinuous but bounded first derivatives even in dimensions n = 3,4. (For n = 5 an example of considered type was constructed by J. Nečas (see [10]).

In the introduction we give a brief survey of the problem of regularity and counterexamples. In Chapter 1 there will be studied the counterexample mentioned above. In Chapter 2 we add seme calculations omitted in Chapter 1 in details.

Key words: Regularity, elliptic systems

Classification: 35J60, 35D10

Introduction. The problem of regularity (or analyticity) of weak solutions of nonlinear elliptic systems can be traced to the beginning of this century - to the 19. D. Hilbert's problem and can be expressed by the question: Supposing a_r^i and u_0 in (1) to be analytic, is the weak solution u also analytic function? The history of this problem is described in several books and papers (see [5],[6],

[4]), hence we will mention here only some crucial points. Very soon - in 1939 - the problem was solved positively for systems of equations of second order in plane by Ch.B. Morrey. Very important further step was made by E.De Giorgi and J. Nash in 1957. They proved regularity of solution of one equation of second order in the space R_n of arbitrarily high dimension n. Another positive result was proved by one of the authors (J. Nečas - in 1967) for equations of arbitrarily high order in plane. Almost immediately there appeared counterexamples (E.De Giorgi - 1968, E. Giusti, M. Miranda - 1968), showing that the situation of one equation of second order in R_n or of systems of arbitrary order in plane is in some sense exceptional and that there exist systems with analytic coefficients whose solutions are not even continuous (x). Unfortunately, these counterexamples have some disadvantages:

(i) They have analytic coefficients, they are naturally defined on Sobolev spaces W_2^1, but the corresponding operators are not differentiable on this space.

(ii) For low dimensions (which play the most important role in physics) it is unclear, whether the irregular solution is unique or if, perhaps, there could exist another regular solution of system in question (xx).

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x) are bounded and have unbounded gradients.

xx) i.e. the typical quasilinear system

$$\sum_{i,j,s=1}^{n} D^i(A_{rs}^{ij}(u)\; D^j u_s) = 0 \text{ for } r = 1,\dots,n.$$

In 1977 J. Nečas constructed a counterexample without these disadvantages and working in all dimensioms $n \geqq 5$, but the problem in $n = 3,4$ still remained unsolved. The aim of this paper is to give a counterexample with unique irregular solution in R_n with dimensions $n \geq 3$.

Chapter 1

1.1. Notation. Let $\Omega = \{x \in R_n; |x| < 1\}$, $u: \Omega \to R_{n^2}$; $u = \{u_{ij}\}_{i,j=1}^n$.

Let us denote

$$D^k u_{ij} = \frac{\partial u_{ij}}{\partial x_k}, \quad \delta_{ij} \text{ the Cronecker symbol,}$$

$$\nabla_j u = \sum_{i=1}^n D^i u_{ji}, \quad \|\nabla u\|^2 = \sum_{j=1}^n (\nabla_j u)^2, \quad (\nabla u, \nabla v) = \sum_{j=1}^n \nabla_j u \, \nabla_j v,$$

for a fixed real number γ let

$$\nabla_{ijk} u = D^k u_{ij} + \gamma (\delta_{ij} \nabla_k u + \delta_{ik} \nabla_j u + \delta_{jk} \nabla_i u),$$

$$\|\delta u\|^2 = \sum_{i,j,k=1}^n (\nabla_{ijk} u)^2, \quad (\delta u, \delta v) = \sum_{i,j,k=1}^n \nabla_{ijk} u \, \nabla_{ijk} v.$$

1.2. System and its solution. Let γ, λ, ν be real numbers. We shall consider the system

$$(2) \quad \sum_{k=1}^n D^k \{ D^k u_{ij} + \gamma (\delta_{ij} \nabla_k u + \delta_{ik} D^j u_{kk}) + \lambda \nabla_i u \, \nabla_j u \, \nabla_k u \, [1 + \|\nabla u\|^2]^{-1} + \delta_{ik} \nabla_j u \{ \gamma (4 + 3\gamma(n+2)) + 3\gamma\lambda \|\nabla u\|^2 [1 + \|\nabla u\|^2]^{-1} + \nu \|\nabla u\|^4 [1 + \|\nabla u\|^2]^{-2} \} \} = 0.$$

We shall prove that the function $u = \{u_{ij}\}_{i,j=1}^n$

$$(3) \qquad u_{ij}(x) = x_i x_j |x|^{-1} - \frac{1}{n} |x| \delta_{ij}$$

is a weak solution of the Dirichlet boundary problem for the system (2), i.e. for every infinitely differentiable function φ with compact support in Ω the equality

$$(4) \quad \langle Au, \varphi \rangle = \int_\Omega \{ (\nabla_{ijk} u + \lambda \nabla_i u \ \nabla_j u \ \nabla_k u [1 + \| \nabla u \|^2]^{-1}) \nabla_{ijk} \varphi + \\ + \nu \| \nabla u \|^4 [1 + \| \nabla u \|^2]^{-2} \ (\nabla u, \nabla \varphi \ ) \} = 0$$

holds, if the numbers λ, ν, γ satisfy the following conditions

$$(5) \quad \lambda = [1 + (n - \frac{1}{n})^2](n - \frac{1}{n})^{-2} \ (\frac{1}{n-1} - \gamma),$$

$$(6) \quad \nu = - (n - \frac{1}{n})^{-5} \{ 3\gamma^2 (n+1)(n - \frac{1}{n}) + \gamma (n^2 + 3n + 2) + \\ + 1 + \frac{1}{n} \} \times [1 + (n - \frac{1}{n})^2]^2 .$$

1.3. Unicity of the solution. Unicity of the solution is an immediate consequence of the following inequality

$$(7) \qquad \langle DA(u) \varphi , \varphi \rangle \geqq C \| \varphi \|^2_{[W^1_2]^{n^2}} \quad ,$$

holding for a positive coonstant C and a class of test functions φ , and which implies that the operator A is strongly monotone. In fact, as it is proved in 2.3, we establish an algebraic condition of monotonicity, i.e. the integrand of $\langle DA(u) \varphi , \varphi \rangle$ is greater than

$$(1 - \frac{3}{4} \ \frac{\lambda^2}{\nu}) \ \| \delta \varphi \|^2 .$$

We have

$$\langle DA(u) \varphi, \varphi \rangle = \int_\Omega \{ \| \delta \varphi \|^2 + \lambda \{ \{ \nabla_i u \ \nabla_j u \nabla_k \varphi + \nabla_i u \ \nabla_j \varphi \ \nabla_k u + \\ + \nabla_i \ \varphi \ \nabla_j u \ \nabla_k u \} [1 + \| \nabla u \|^2]^{-1} + \nabla_i u \ \nabla_j u \ \nabla_k u \ (-2 (\nabla u, \nabla \varphi \ )$$

$[1 + \|\nabla u\|^2]^{-2}\} \nabla_{ijk}\varphi + \nu\{\{4\|\nabla u\|^2(\nabla u,\nabla\varphi)^2 +$

$+ \|\nabla u\|^4 \|\nabla\varphi\|^2\}[1 + \|\nabla u\|^2]^{-2} -$

$- 4 \|\nabla u\|^4(\nabla u, \nabla\varphi)^2[1 + \|\nabla u\|^2]^{-3}\}\}.$

Estimating the second term by Hölder inequality we get

$$(8) \qquad \langle DA(u)\varphi, \varphi\rangle \geqq \int_\Omega (1 - \frac{3\lambda^2}{4\nu}) \|\delta\varphi\|^2.$$

Taking in consideration the symmetry of the solution and of the system in the indices i, j, we can choose the test function φ symmetric in i, j, too. Thus it suffices to consider the test functions φ with only $m = \frac{1}{2} n (n + 1)$ different components. For such functions φ we get (supposing that $\gamma < 0$)

$$(9) \qquad \| \|\delta\varphi\| \|^2_{L_2} \geqq (1 - \frac{n\gamma}{4 + 3\gamma(n+2)}) \|\varphi\|^2_{[\overset{\circ}{W}{}^1_2]^{n^2}} .$$

Summarizing the inequalities (8) and (9) we obtain (7) with a constant C which is positive if

$1 - \frac{n\gamma}{4 + 3\gamma(n+2)}) > 0$, which implies the inequality $\gamma < \frac{-2}{n + 3}$;

and if

$$1 - \frac{3\lambda^2}{4\nu} > 0.$$

The second condition implies that $\gamma \in (\gamma_1, \gamma_2)$ where (for $n = 3$)

$$\gamma_i = \frac{-27 \pm 2\sqrt{42}}{102}$$

is approximately $\gamma_1 = -0{,}39$, $\gamma_2 = -0{,}13$. Analogous numerical results show that the counterexample works in dimensions $n = 4,5$. For higher dimensions the function

$$\gamma_1(n) + \frac{2}{n+3}$$

is a decreasing function of variable n. It proves that for all $n \geq 3$ we can choose γ so that the function u given by (3) is the unique solution of the system (2) with analytic coefficients and linear growth. Moreover, u is the solution of the Dirichlet boundary problem with analytic boundary condition u_o.

Chapter 2

2.1. Deduction of $\lambda(\gamma)$ and $\nu(\gamma)$. The system (4) can be written in the form

$$(10) \qquad \int \{\Phi_{ijk} \nabla_{ijk}\varphi + \Psi_i \nabla_i \varphi\}\, dx = 0, \quad \varphi_{ij} = \varphi_{ji} \in \mathcal{D}.$$

By means of the Gauss formula we deduce from (10) the system in a strong form

$$(11) \qquad D^k \Phi_{ijk} + \gamma D^j(\Phi_{kki} + \Phi_{kik} + \Phi_{ikk}) + D^j \Psi_i = 0, \qquad (i,j = 1,\dots,n),$$

remembering that

$$(12) \qquad \begin{aligned} \Phi_{ijk} &= \nabla_{ijk}u + \lambda[1 + \|\nabla u\|^2]^{-1} \nabla_i u\, \nabla_j u\, \nabla_k u, \\ \Psi_i &= \nu[1 + \|\nabla u\|^2]^{-2} \|\nabla u\|^4\, \nabla_i u. \end{aligned}$$

We want to choose the parameters ν, λ, γ in such a way that the function

$$(13) \qquad u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{1}{n}\delta_{ij}|x|, \quad (i,j = 1,\dots,n)$$

would be the solution of (11).

After tedious but not difficult calculations we get the following expressions for Φ and Ψ (see (12)) in case of function u given by (13):

$$\text{(14)}\quad \begin{aligned} \Phi_{ijk} &= |x|^{-1}[a(\delta_{ik}x_j + \delta_{jk}x_i) + b\,\delta_{ij}x_k] + c\,\frac{x_i x_j x_k}{|x|^3}, \\ \Psi_i &= (n - \tfrac{1}{n})^5[1 + (n - \tfrac{1}{n})^2]^{-2}\,\frac{x_i}{|x|}, \end{aligned}$$

where

$$\text{(15)}\quad a = 1 + \gamma(n - \tfrac{1}{n}),\ b = -\tfrac{1}{n} + \gamma(n - \tfrac{1}{n}),$$
$$c = \lambda[1 + (n - \tfrac{1}{n})^2]^{-1}(n - \tfrac{1}{n})^3 - 1.$$

Substituting (14) and (15) into (11) and differentiating we put the coefficients of $\delta_{ij}|x|^{-1}$ and $x_i x_j |x|^{-3}$ equal zero. Thus we obtain

$$\text{(16)}\quad \begin{aligned} &2a + (n-1)b + \nu[1 + (n - \tfrac{1}{n})^2]^{-2}\,(n - \tfrac{1}{n})^5 + \\ &+ \gamma[2(2 + n)a + (2 + n)b + 3c] = 0, \\ &-2a + (n-1)c - \nu[1 + (n - \tfrac{1}{n})^2]^{-2}\,(n - \tfrac{1}{n})^5 - \\ &- \gamma[2(2 + n)a + (2 + n)b + 3c] = 0. \end{aligned}$$

From here it follows immediately that

$$\text{(17)}\quad \lambda = [1 + (n - \tfrac{1}{n})^2]\,(n - \tfrac{1}{n})^{-2}\left(\frac{1}{n-1} - \gamma\right),$$

$$\text{(18)}\quad \nu = -(n - \tfrac{1}{n})^{-5}\{3\gamma^2(n+1)(n - \tfrac{1}{n}) + \gamma(n^2 + 3n + 2) + $$
$$+ (1 + \tfrac{1}{n})\} \times [1 + (n - \tfrac{1}{n})^2]^2 .$$

2.2. Equivalent norms. We are to formulate sufficient conditions on parameter γ under which there exists a constant $c_\gamma > 0$ such that

$$\text{(19)}\quad \|\delta u\|^2 \geqq c_\gamma\, \|Du\|^2$$

where $\|Du\|^2 = \sum_{i,j,k} (D^k u_{ij})^2$, $\|\delta u\|^2 = \sum_{i,j,k} (\nabla_{ijk} u)^2$.

It is

$$\|\delta u\|^2 = \sum_{i,j,k} [D^k u_{ij} + \gamma(\delta_{ij} \nabla_k u + \delta_{ik} \nabla_j u + \delta_{jk} \nabla_i u)]^2 \geq$$

$$\geq \text{(supposing that } \gamma < 0) \geq \|Du\|^2 + 2\gamma \sum_{i,k} |D^k u_{ii}|\, |\nabla_k u| +$$

$$+ [4\gamma + 3\gamma^2(n+2)]\ \|\nabla u\|^2 \geq \|Du\|^2 + 2\gamma \sqrt{n}\ \|Du\|\ \|\nabla u\| +$$

$$+ [4\gamma + 3\gamma^2(n+2)]\ \|\nabla u\|^2 = [4\gamma + 3\gamma^2(n+2)]\ \{\|\nabla u\| +$$

$$+ (\gamma\sqrt{n}\ \|Du\|)[4\gamma + 3\gamma^2(n+2)]^{-1}\}^2 +$$

$$+ (1 - (\gamma n)[4 + 3\gamma(n+2)]^{-1})\ \|Du\|^2.$$

It is easy to see that for

$$\gamma < -\frac{2}{n+3} \tag{20}$$

it takes place (19) with $c_\gamma = 1 - (\gamma n)[4 + 3\gamma(n+2)]^{-1} > 0$.

2.3. Monotonicity condition. Let us suppose that $\lambda > 0$, $\nu > 0$. (From (17) it follows that $\lambda > 0$ is implied by the condition (20).) Putting

$$\overline{\nabla_i u} = \nabla_i u(1 + \|\nabla u\|^2)^{\frac{1}{2}} \tag{21}$$

and denoting by $I(\varphi)$ the integrand of $\langle DA(u)\varphi, \varphi\rangle$ we have

$$I(\varphi) = \|\delta\varphi\|^2 + \tag{22}$$

$$+ \lambda\{[\overline{\nabla_i u}\ \overline{\nabla_j u}\ \nabla_k\varphi + \overline{\nabla_i u}\ \nabla_j\varphi\ \overline{\nabla_k u} + \nabla_i\varphi\ \overline{\nabla_j u}\ \overline{\nabla_k u} -$$

$$- 2(\overline{\nabla u}, \nabla\varphi)\ \overline{\nabla_i u}\ \overline{\nabla_j u}\ \overline{\nabla_k u}]\ \nabla_{ijk}\varphi\} + \nu\{4\ \|\overline{\nabla u}\|^2(\overline{\nabla u}, \nabla\varphi)^2 +$$

$$+ \|\overline{\nabla u}\|^4\ \|\nabla\varphi\|^2 - 4\cdot\|\overline{\nabla u}\|^4(\overline{\nabla u}, \nabla\varphi)^2\}.$$

The expression standing by λ in figure brackets can be estimated by means of Hölder inequality as follows:

$$|\{[\overline{\nabla_i u}\ \overline{\nabla_j u}\ \nabla_k\varphi + \ldots]\ \nabla_{ijk}\varphi\}| \leqq \{\sum_{i,j,k}[\overline{\nabla_i u}\ \overline{\nabla_j u}\ \nabla_k\varphi + \ldots]^2\}^{\frac{1}{2}} = \|\delta\varphi\|\,\{3\ \|\overline{\nabla u}\|^4\ \|\nabla\varphi\|^2 + 2\ \|\overline{\nabla u}\|^2 (\overline{\nabla u}, \nabla\varphi)^2[3 - 6\ \|\overline{\nabla u}\|^2 + 2\ \|\overline{\nabla u}\|^4]\}^{\frac{1}{2}} \leqq$$

(using the fact that $0 \leqq \|\overline{\nabla u}\| < 1$)

$$\leqq \|\delta\varphi\|\,\{3\ [\|\overline{\nabla u}\|^4\ \|\nabla\varphi\|^2 + 2\ \|\overline{\nabla u}\|^2(\overline{\nabla u}, \nabla\varphi)^2(1 - \|\overline{\nabla u}\|^2)]\}^{\frac{1}{2}}.$$

Using the estimate in (22) and putting

$$Q^2 = \|\overline{\nabla u}\|^4\ \|\nabla\varphi\|^2 + 2\ \|\overline{\nabla u}\|^2(\overline{\nabla u}, \nabla\varphi)^2(1 - \|\overline{\nabla u}\|^2)$$

we obtain

$$(23)\qquad I(\varphi) \geqq \|\delta\varphi\|^2 - \sqrt{3}\,\lambda Q\,\|\delta\varphi\| + \nu Q^2 \geqq \|\delta\varphi\|^2\left(1 - \frac{3\lambda^2}{4\nu}\right).$$

Let now

$$(24)\qquad 4\nu > 3\lambda^2$$

and let (20) hold. Then $I(\varphi) \geqq c^*_\gamma\ \|D\varphi\|^2$ with $c^*_\gamma > 0$ and so the monotonicity of the operator A defined by (4) takes place.

References

[1] De GIORGI E.: Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolari, Mem. Acad. Sci. Torino 3(1957), 25-43, Matematika 4(1960), 25-38.

[2] De GIORGI E.: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, Boll. Unione Mat. Italiana 1(1968), 135-138.

[3] GIUSTI E., MIRANDA M.: Un esempio di soluzioni discontinue per un problema di minimo relativo ad un integrale regolare del calcolo delle variazioni, Boll. Unione Mat. Italiana (4)1(1968),219-227.

[4] KOŠELEV A.I.: Regularnosť rešenij kvasilinějnych elliptičeskich sistem, Uspechi mat. nauk 23(1978), 3-49.

[5] LADYŽENSKAJA O.A., URAL'CEVA N.N.: Linějnyje i kvasilinějnyje uravněnija eliptičeskogo tipa, Nauka, Moskva 1964.

[6] MORREY Ch.B.: Differentiability theorems for weak solutions of nonlinear elliptic differential equations, Bull. Am. Math. Soc. 75(1969), 684-705.

[7] MORREY Ch.B.: Existence and differentiability theorems for the solutions of variational problems for multiple integrals, Bull. Am. Math. Soc. 46 (1940), 439-458.

[8] NASH J.: Continuity of solutions of parabolic and elliptic equations, Amer. J. Math. 80(1950),931-954.

[9] NEČAS J.: Sur la régularité des solutions variationelles des équations elliptiques non-linéaires d'ordre 2k en deux dimensions, Annali Sc. Norm. Sup. Pisa 21(1967), 427-457.

[10] NEČAS J.: Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Phys. Tech.,1977, 1N,197-206.

Matemat. fyz. fakulta
Universita Karlova
Malostran. nám. 1
11000 Praha 1
Československo

Matemat. fyz. fakulta
Universita Karlova
Sokolovská 83
18600 Praha 8
Československo

Mat. fyz. fakulta
Universita Karlova
Sokolovská 83
18600 Praha 8
Československo

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