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TENSOR PRODUCTS IN THE CATEGORY OF TOPOLOGICAL SPACES

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Abstract: The category of topological spaces is known to be a closed category. We prove that there is (up to isomorphism) precisely one structure of closed category on the category of topological spaces and also on the category of T_0-spaces.

Key words: Closed category, tensor product, uniform filter, ultraspace, coreflective subcategory.

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Introduction. The category $\mathcal{T}$ of all topological spaces and continuous maps is well known to be a closed category, namely for arbitrary topological spaces X, Y the tensor product $X \otimes Y$ is obtained by proving the set $X \times Y$ with the "topology of separate continuity" and $\mathcal{T}(Y,Z)$ equipped with the topology of pointwise convergence is the value of the corresponding internal hom functor $[-,-]$ at (Y,Z) ($f \otimes g = f \times g$, $[g,h](t) = h \circ t \circ g$). In this paper we shall prove that ($\otimes$, $[-,-]$) is (up to isomorphism) the only structure of closed category on the category $\mathcal{T}$ and also on the category $\mathcal{T}_0$ of all T_0-spaces.

1. <u>Preliminaries and notations</u>. We shall always use the following notations:

$\mathcal{A}(X,Y)$ denotes the set of all $\mathcal{A}$-morphisms $X \longrightarrow Y$. C_2 denotes the Sierpinski doubleton on the set $\{0,1\}$ where $c\ell\{0\} = \{0\}$, $c\ell\{1\} = \{0,1\}$. The forgetful functor $\mathcal{T} \longrightarrow$ Set is denoted by U. We shall often write X instead of UX. If A, B are sets, $M \subset A \times B$, $a \in A$ and $b \in B$, then $aM = \{y \in B : (a,y) \in M\}$ and $Mb = \{x \in A : (x,b) \in M\}$. Let A, B, C be sets, $f: A \times B \longrightarrow C$ a map. Then f^* is the map $A \longrightarrow C^B$ given by $f^*(a)(b) = f(a,b)$ for all $a \in A$, $b \in B$. If $g: A \longrightarrow C^B$ is a map, then g_* is the map $A \times B \longrightarrow C$ given by $g_*(a,b) = g(a)(b)$ for all $a \in A$, $b \in B$.

Let X, Y be topological spaces. Then the topology of the space $X \otimes Y$ i.e. the topology of separate continuity τ on $UX \times UY$ is defined as follows: τ is the initial topology with respect to the class $\mathcal{S}_{XY}$ of all maps $f: UX \times UY \longrightarrow UZ$, $Z \in \mathcal{T}$, such that $f(a,-): Y \longrightarrow Z$ and $f(-,b): X \longrightarrow Z$ are continuous maps for each $a \in X$, $b \in Y$. Equivalently, τ is the initial topology with respect to the set of all maps $f: UX \times UY \longrightarrow UC_2$ belonging to $\mathcal{S}_{XY}$.

The notion of closed category is used in the sense of [7; p. 180] and it coincides with the notion of symmetric monoidal closed category used in [3]. Recall that a triple $(\mathcal{A}, \Box, H)$ is said to be a closed category provided that $(\mathcal{A}, \Box)$ is a symmetric monoidal category [7; p. 180], $H: \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a functor (called an internal hom functor) and there exists a natural equivalence $\gamma = (\gamma_{ABC}): \mathcal{A}(A \Box B, C) \longrightarrow \mathcal{A}(A, H(B,C))$. A tensor product is a symmetric monoidal structure extendable to a structure of closed category (= closed structure).

Cardinals are initial ordinals where each ordinal is the set of its predecessors.

Any coreflective subcategory of $\mathcal{T}$ and $\mathcal{T}_0$ (see [4]) is supposed to be full and isomorphism-closed. If $\mathcal{B} \in \{\mathcal{T}, \mathcal{T}_0\}$ and $\mathcal{A}$ is a class of $\mathcal{B}$-objects or a subcategory of $\mathcal{B}$, then the object class of the coreflective hull of $\mathcal{A}$ in $\mathcal{B}$ consists precisely of $\mathcal{B}$-extremal quotients of $\mathcal{B}$-coproducts of objects belonging to $\mathcal{A}$. Recall that any non-trivial coreflective subcategory of the category $\mathcal{B} \in \{\mathcal{T}, \mathcal{T}_0\}$ is bicoreflective, i.e. coreflections are modifications (see [4]).

2. <u>Closed structures on the category</u> $\mathcal{T}$. The following theorem considerably simplifies the study of closed structures on $\mathcal{T}$. Recall (see [7; p. 26]) that a concrete category is a pair $(\mathcal{K}, V)$ where $\mathcal{K}$ is a category and $V: \mathcal{K} \longrightarrow$ Set is a faithful functor.

2.1. <u>Theorem</u> [8]. Let $(\mathcal{K}, V)$ be a concrete category with the following properties:

(1) For every constant map $c: VA \longrightarrow VB$ there exists a $\mathcal{K}$-morphism $k: A \longrightarrow B$ with $Vk = c$.

(2) For every bijection $f: VA \longrightarrow X$ there exists a $\mathcal{K}$-isomorphism $s: A \longrightarrow B$ with $Vs = f$.

(3) There exists a $\mathcal{K}$-object A with card $VA \geqq 2$.

If there is a closed structure $(\circ, G)$ on $\mathcal{K}$, then there exists a closed structure $(\Box, H)$ on $\mathcal{K}$ isomorphic with $(\circ, G)$ with the following properties:

(a) Card $VI = 1$ where I is a unit of $\Box$.

(b) $VA \times VB \subset V(A \Box B)$,

(c) for any $r, s: A \Box B \longrightarrow C$, $Vr|VA \times VB = Vs|VA \times VB$

implies $r = s$,

(d) $V(f \square g) | VA \times VB = Vf \times Vg$,

(e) $VH(B,C) = \mathcal{K}(B,C)$,

(f) if $\gamma : \mathcal{K}(A \square B,C) \longrightarrow \mathcal{K}(A,H(B,C))$ is the natural equivalence corresponding to $(\square ,H)$, then $V\gamma(r) = (Vr)^*$ and $V\gamma^{-1}(s) = (Vs)_*$ for arbitrary $\mathcal{K}$-objects A, B, C and $\mathcal{K}$-morphisms $f:A \longrightarrow A'$, $g:B \longrightarrow B'$.

If, moreover, $\mathcal{K}$ satisfies

(4) $X \subset VA$ implies that there exists a $\mathcal{K}$-morphism $j: :B \longrightarrow A$ such that $VB = X$ and $Vj(x) = x$ for each $x \in X$,

(5) for every $\mathcal{K}$-epimorphism g Vg is a surjection,

then

(g) $VA \times VB = V(A \square B)$ for any $\mathcal{K}$-objects A, B.

The category $\mathcal{T}$ fulfils (1) - (5) of 2.1 so that without loss of generality we can adopt:

2.2. <u>Convention</u>. All closed structures on $\mathcal{T}$ will be assumed to satisfy (a) - (g) of 2.1.

It is obvious that a closed structure $(\square ,H)$ on $\mathcal{T}$ satisfying (a) - (g) of 2.1 has also the following property:

(h) The natural isomorphisms $r_X : X \square \{*\} \longrightarrow X$, $l_X : \{*\} \square X \longrightarrow X$ and the symmetry $c_{XY} : X \square Y \longrightarrow Y \square X$ corresponding to $\square$ are given by $(x,*) \longmapsto x$, $(*,x) \longmapsto x$ and $(x,y) \longmapsto (y,x)$ respectively for any topological spaces X, Y.

If $(\square ,H)$ is a closed structure on $\mathcal{T}$, then the tensor product $\square$ preserves $\mathcal{T}$-coproducts and $\mathcal{T}$-extremal epimorphisms (which coincide with the regular ones in $\mathcal{T}$). Therefore if $\mathcal{A}$ is a class of topological spaces such that the coreflective hull of $\mathcal{A}$ coincides with $\mathcal{T}$, then any tensor product (more exactly its object function) is uniquely determined by its values on $\mathcal{A} \times \mathcal{A}$.

It is obvious that the coreflective hull of the class of all ultraspaces in $\mathcal{T}$ coincides with $\mathcal{T}$.

2.3. Definition [2]. A filter $\mathcal{F}$ on a set A is said to be uniform provided that for all $F \in \mathcal{F}$ card F = card A.

By [2], if $\mathcal{U}$ is an ultrafilter on a set B, then there exists a uniform ultrafilter $\mathcal{V}$ on a set A and a surjective map $f:A \longrightarrow B$ such that $\mathcal{U} = \{f[V]:V \in \mathcal{V}\}$. In fact, if $\mathcal{U}$ is principal, then it is evident. If $\mathcal{U}$ is a non principal ultrafilter, then take an arbitrary uniform ultrafilter $\mathcal{W}$ on B. Then $\mathcal{U} \cdot \mathcal{W}$ (see [2; p. 156]) is a uniform ultrafilter on $B \times B$ (see [2; 7.21(a), 7.20(c)]) and $\mathcal{U} = \{p_1[V]: V \in \mathcal{U} \cdot \mathcal{W}\}$ where $p_1 : B \times B \longrightarrow B$; $(x,y) \longmapsto x$ is a projection (see [2; 7.21(b) and 7.19(a)]). Hence, any ultraspace is an extremal quotient of a uniform ultraspace (an ultraspace is said to be uniform provided that its corresponding ultrafilter is uniform) so that the coreflective hull of the class of all uniform ultraspaces in $\mathcal{T}$ coincides with $\mathcal{T}$. Denote by $\mathcal{L}$ the class of all uniform ultraspaces defined on cardinals. (Let α be an infinite cardinal, $\mathcal{U}$ a uniform ultrafilter on α . Then the corresponding ultraspace is defined on $\alpha + 1$ as follows: $\{x\}$ is open for all $x \in \alpha$ and $\{V \cup \{\alpha\} : V \in \mathcal{U}\}$ is the family of all neighbourhoods of α .) Then we have:

2.4. Proposition. Any tensor product $\square$ on $\mathcal{T}$ is uniquely determined by its values on $\mathcal{L} \times \mathcal{L}$.

Let A be an infinite set and $\mathcal{F}$ a free filter on A (i.e.

$\cap \mathcal{F} = \emptyset$). Let $a \notin A$. Define the topology on $A \cup \{a\}$ in the following way: $V \subset A \cup \{a\}$ is open if and only if $V \subset A$ or $a \in V$ and $V - \{a\} \in \mathcal{F}$. Such topological spaces we shall call filter spaces and denote by $(A,a,\mathcal{F})$ or only by (A,a).

2.5. Proposition. Let (A,a), (B,b) be filter spaces, $c\ell$, $c\ell_A$, $c\ell_B$ closure operations of the spaces $(A,a) \otimes (B,b)$, (A,a), (B,b) respectively and $M \subset (A \cup \{a\}) \times \times (B \cup \{b\})$. Then

(i) If $(x,y) \in A \times B$, then $(x,y) \in c\ell M$ if and only if $(x,y) \in M$.

(ii) If $y \in B$, then $(a,y) \in c\ell M - M$ if and only if $a \in c\ell_A My$.

(iii) If $x \in A$, then $(x,b) \in c\ell M - M$ if and only if $b \in c\ell_B xM$.

(iv) $(a,b) \in c\ell M$ if and only if $(a,b) \in M$ or $a \in c\ell_A Mb$ or $b \in c\ell_B aM$ or $a \in c\ell_A C$ where $C = \{x \in A : b \in c\ell_B xM\}$ or $b \in c\ell_B D$ where $D = \{y \in B : a \in c\ell_A My\}$.

Proof. Easy to check.

It is easy to see that if $(\square, H)$ is a closed structure on $\mathcal{T}$, then for arbitrary spaces X, Y, $X \otimes Y \xrightarrow{id_{X \times Y}} X \square Y$ is a continuous map (it is evidently separately continuous). Obviously, the projections $p_1 : X \square Y \xrightarrow{1 \square c} X \square \{*\} \xrightarrow{r_X} X$; $(x,y) \mapsto x$, $p_2 : X \square Y \xrightarrow{k \square 1} \{*\} \square Y \xrightarrow{l_Y} Y$; $(x,y) \mapsto y$ are continuous so that $id_{UX \times UY} : X \square Y \to X \times Y$ is a continuous map. Hence $X \otimes Y \leqq X \square Y \leqq X \times Y$ for all spaces X, Y, where $(X, c\ell_X) \leqq (Y, c\ell_Y)$ if and only if $X = Y$ and $c\ell_X M \subset c\ell_Y M$ for each $M \subset X$ ($X < Y$ if and only if $X \leqq Y$ and $X \neq Y$), and then, evidently, $H(X,Y) \leqq [X,Y]$ for all $X, Y \in \mathcal{T}$.

Let now (A,a), (B,b) be filter spaces and $(x,y) \in$ $\in ((A \cup \{a\}) \times (B \cup \{b\})) - \{(a,b)\}$. Then $(x,y) \in c\ell M$ in $(A,a) \otimes (B,b)$ if and only if $(x,y) \in c\ell M$ in $(A,a) \times (B,b)$. Hence we obtain

2.6. <u>Lemma</u>. $(A,a) \otimes (B,b) < (A,a) \boxdot (B,b)$ ($\leq (A,a) \times$ $\times (B,b)$) for a tensor product $\boxdot$ on $\mathcal{T}$ if and only if there exists $M \subset (A \cup \{a\}) \times (B \cup \{b\})$ with $(a,b) \in c\ell M$ in $(A,a) \boxdot (B,b)$ and $(a,b) \notin c\ell M$ in $(A,a) \otimes (B,b)$.

Let α be an infinite cardinal and $A \subset \alpha \times \alpha$ a symmetric reflexive relation on α such that for each $x \in \alpha$ card $xA \prec \alpha$. Define the α-sequence $a: \alpha \longrightarrow \alpha$ as follows: $a_0 = 0$; let $M_t = \{x \in \alpha$: there exists $y \in \alpha$, $y \leq a_t$ such that $(x,y) \in A\}$. Then a_{t+1} is the smallest element $x \in \alpha$ with $M_t \subset x$. If $t \in \alpha$ is a limit ordinal, then $a_t = \sup \{a_x : x < t\}$. Obviously, $(a_x)_{x \in \alpha}$ is an increasing α-sequence. Put $R_x =$ $= [a_x, a_{x+1}) = \{y \in \alpha : a_x \leq y < a_{x+1}\}$. Then we have:

2.7. <u>Lemma</u>. If $(R_x \times R_y) \cap A \neq \emptyset$, then $x = y$ or $x = y + 1$ or $y = x + 1$.

Proof. Let $x < y$ and $(b,c) \in (R_x \times R_y) \cap A$. Since $b \in R_x$ $b < a_{x+1}$ and then $c \in \{z \in \alpha$: there exists $t < a_{x+1}$ with $(t,z) \in A\}$, i.e. $c < a_{x+2}$. Hence $a_{y+1} \leq a_{x+2}$ so that $y \leq x + 1$. If $y < x$, then we consider $(b,c) \in (R_y \times R_x) \cap A$ (A is symmetric so that $(R_y \times R_x) \cap A$ is non empty).

Let now α be an infinite cardinal and $\mathcal{F}$ the generalized Fréchet filter on α ($A \in \mathcal{F}$ if and only if card $(\alpha - A) <$ $< \alpha$). Denote by $C(\alpha)$ the corresponding filter space defined on $\alpha + 1$. Let $\boxdot$ be a tensor product on $\mathcal{T}$ with $C(\alpha) \boxdot C(\alpha) > C(\alpha) \otimes C(\alpha)$. Then by 2.6 there exists

$M \subset (\alpha + 1) \times (\alpha + 1)$ for which $(\alpha, \alpha) \in c\ell_\square M - c\ell_\otimes M$ ($c\ell_\square$, $c\ell_\otimes$ are the closure operations of $C(\alpha) \square C(\alpha)$ and $C(\alpha) \otimes C(\alpha)$ respectively). It is easy to see that then αM and $M\alpha$ are closed in $C(\alpha)$ and therefore $\{\alpha\} \times \alpha M$ and $M\alpha \times \{\alpha\}$ are closed in $C(\alpha) \square C(\alpha)$. Hence $(\alpha, \alpha) \in c\ell_\square M' - c\ell_\otimes M'$ where $M' = M \cap (\alpha \times \alpha)$. Since $\square$ is symmetric $(\alpha, \alpha) \in c\ell_\square M' - c\ell_\otimes M'$ if and only if $(\alpha, \alpha) \in c\ell_\square (M' \cup (M')^{-1}) - c\ell_\otimes (M' \cup (M')^{-1})$. Thus we obtain:

2.8. Lemma. If $\square$ is a tensor product on $\mathcal{T}$, then $C(\alpha) \otimes C(\alpha) < C(\alpha) \square C(\alpha)$ if and only if there exists a symmetric subset $M \subset \alpha \times \alpha$ (i.e. $M = M^{-1}$) with $(\alpha, \alpha) \in c\ell_\square M - c\ell_\otimes M$.

2.9. Proposition. Let $(\square, H)$ be a closed structure on $\mathcal{T}$ and α an infinite cardinal. If $C(\alpha) \square C(\alpha) \neq C(\alpha) \otimes C(\alpha)$, then $(\alpha, \alpha) \in c\ell_\square \Delta_\alpha$ ($\Delta_\alpha = \{(x,x) : x \in \alpha\}$, $c\ell_\square$, $c\ell_\otimes$ are closure operations of $C(\alpha) \square C(\alpha)$ and $C(\alpha) \otimes C(\alpha)$ respectively).

Proof. Let $C(\alpha) \square C(\alpha) \neq C(\alpha) \otimes C(\alpha)$. Then by 2.8 there exists a symmetric subset $M' \subset \alpha \times \alpha$ with $(\alpha, \alpha) \in c\ell_\square M' - c\ell_\otimes M'$. Since $(\alpha, \alpha) \notin c\ell_\otimes M'$, the set $A = \{x \in \alpha : \alpha \in c\ell\, xM' = \{x \in \alpha : \alpha \in c\ell\, M'x\}$ is closed in $C(\alpha)$ so that $(\alpha, \alpha) \in c\ell_\square M'' - c\ell_\otimes M''$ where $M'' = M' - ((\bigcup_{x \in A} (\{x\} \times (xM'))) \cup (\bigcup_{x \in A} (M'x) \times \{x\})))$. Hence for each $x \in \alpha$ card $xM'' < \alpha$.

Suppose $(\alpha, \alpha) \notin c\ell_\square \Delta_\alpha$. Put $M = \Delta_\alpha \cup M''$. Then $(\alpha, \alpha) \in c\ell_\square M - c\ell_\otimes M$ and M is reflexive symmetric relation on $\alpha \times \alpha$ with card $xM < \alpha$ for each $x \in \alpha$. Put $E = \bigcup_{x \in \alpha} (R_x \times R_x)$ (see 2.7). Then E is an equivalence relation on α . Denote by e the natural projection $\alpha \longrightarrow \alpha/E$. Define $e': \alpha + 1 \longrightarrow (\alpha/E \cup \{\alpha\})$ by $\alpha \longmapsto \alpha$, $e'|\alpha = e$. If $C'(\alpha)$ is an extremal quotient space determined by the map $e': C(\alpha) \longrightarrow (\alpha/E \cup \{\alpha\})$, then $C'(\alpha)$ is isomorphic with $C(\alpha)$. The map $e' \square e': C(\alpha) \square C(\alpha) \longrightarrow C'(\alpha) \square C'(\alpha)$ is continuous and the set $\tilde{M} = (e' \square e')[M]$ has the following property: For each $\bar{x} \in \alpha/E$ $\bar{x}\tilde{M} \subset \{\overline{x-1}, \bar{x}, \overline{x+1}\}$ if $\bar{x} = \overline{y+1}$ and $\bar{x}\tilde{M} \subset \{\bar{x}, \overline{x+1}\}$ if x is a limit ordinal where $\bar{x} = R_x$ for each $x \in \alpha$. Since $(\alpha,\alpha) \in cl_{\square} M$, $(\alpha,\alpha) \in cl\, \tilde{M}$ in $C'(\alpha) \square C'(\alpha)$. But $\tilde{M} = M_1 \cup M_2 \cup \Delta_{\alpha/E}$ where $M_1 \subset \{(\overline{x+1}, \bar{x}): :\bar{x} \in \alpha/E\}$, $M_2 = \{(\bar{x}, \overline{x+1}): \bar{x} \in \alpha/E\}$ and this implies that $(\alpha,\alpha) \in cl\, \Delta_{\alpha/E}$ in $C'(\alpha) \square C'(\alpha)$ - a contradiction.

The filter $\mathcal{F}$ on α corresponding to $C(\alpha)$ is the intersection of all uniform ultrafilters on α (see [2]). Therefore $C(\alpha)$ is an extremal quotient of the $\mathcal{T}$-coproduct of the family $\mathcal{S}_\alpha$ of all uniform ultraspaces on $\alpha + 1$ (corresponding to all uniform ultrafilters on α) and the map e: $: \bigsqcup_{S \in \mathcal{S}_\alpha} S \longrightarrow C(\alpha)$ with $e|S = 1_{\alpha+1}$ for all $S \in \mathcal{S}_\alpha$ is an extremal epimorphism. Let $C(\alpha) \square C(\alpha) \neq C(\alpha) \otimes C(\alpha)$. Since $1 \square e: C(\alpha) \square (\bigsqcup_{S \in \mathcal{S}_\alpha} S) = \bigsqcup_{S \in \mathcal{S}_\alpha} (C(\alpha) \square S) \longrightarrow C(\alpha) \square C(\alpha)$ is an extremal epimorphism there exists $T \in \mathcal{S}_\alpha$ with $(\alpha,\alpha) \in \in cl\, \Delta_\alpha$ in $C(\alpha) \square T$ (because $(\alpha,\alpha) \in cl_{\square} \Delta_\alpha$ in $C(\alpha) \square C(\alpha)$). Consider the bijection $\mathcal{T}(C(\alpha) \square T, C_2) \longrightarrow \longrightarrow \mathcal{T}(C(\alpha), H(T, C_2))$; $t \longmapsto t^*$. Since the map $f: C(\alpha) \square T \rightarrow \longrightarrow C_2$; $f[\Delta_\alpha] = \{0\}$, $f(x,y) = 1$ otherwise is not continuous ($(\alpha,\alpha) \in cl\, \Delta_\alpha$) the corresponding map $f^*: C(\alpha) \longrightarrow H(T, C_2)$ is not continuous (it is easy to see that f^* is a map $C(\alpha) \rightarrow \longrightarrow H(T, C_2)$). Hence there exists a set $K \subset C(\alpha)$ with

$\alpha \in c\ell K$ in $C(\alpha)$ and $f^*(\alpha) \notin c\ell f^*[K]$. Let S be an arbitrary non principal ultraspace on $\alpha + 1$ for which K is a member of its corresponding ultrafilter and $S \neq T$. Then $f^* : S \longrightarrow H(T, C_2)$ is not continuous. But the bijection $\mathcal{T}(S \,\square\, T, C_2) \longrightarrow \mathcal{T}(S, H(T, C_2));\ t \longmapsto t^*$ implies that $f: S \,\square\, T \longrightarrow C_2$ is not continuous so that (one can easily see) $(\alpha, \alpha) \in c\ell\, \Delta_\alpha$ in $S \,\square\, T$. Evidently, $S \neq T$ implies $(\alpha, \alpha) \notin c\ell\, \Delta_\alpha$ in $S \times T$ so that $S \,\square\, T \not\leq S \times T$ - a contradiction.

Thus we have proved:

2.10. <u>Proposition</u>. If $(\square, H)$ is a closed structure on $\mathcal{T}$, then for any infinite cardinal α $C(\alpha) \,\square\, C(\alpha) = C(\alpha) \otimes C(\alpha)$.

2.11. <u>Lemma</u>. If D is a discrete space and $(\square, H)$ a closed structure on $\mathcal{T}$, then for any space Y $H(D,Y) = [D,Y]$.

Proof. Immediate from the fact that $X \,\square\, D = \coprod_{d \in D} (X \,\square\, \{d\})$ for any space X.

Denote by $\mathcal{T}_\alpha$ the coreflective hull of the space $C(\alpha)$ in $\mathcal{T}$. Evidently, X belongs to $\mathcal{T}_\alpha$ if and only if the topology of X is determined by a convergence of α-sequences. Clearly, C_2 belongs to $\mathcal{T}_\alpha$ and $\mathcal{T}_\alpha$ is closed under the formation of subspaces. Therefore if M is a subspace of the space X and $X' \xrightarrow{id_{UX}} X$, $M' \xrightarrow{id_{UM}} M$ are the $\mathcal{T}_\alpha$-coreflections of X, M respectively, then M' is the subspace of X' on the subset UM.

Let X be a topological space such that $C(\alpha) \,\square\, X = C(\alpha) \otimes X$. Then, obviously, $\mathcal{T}(C(\alpha), H(X, C_2)) = \mathcal{T}(C(\alpha), [X, C_2])$. Denote by $H_\alpha(X, C_2)$ the $\mathcal{T}_\alpha$-coreflection of $H(X, C_2)$. Since $H(X, C_2) \leq [X, C_2]$ and $\mathcal{T}(C(\alpha), H(X, C_2)) = \mathcal{T}(C(\alpha), [X, C_2])$, $H_\alpha(X, C_2)$ is also a $\mathcal{T}_\alpha$-coreflection of $[X, C_2]$. One can easily see that the family $\mathcal{B}^{\alpha}_X$ of all sets $\prod_{x\in X} V_x$ where V_x are open subsets of C_2 and card $\{x\in X: V_x = \{1\}\} < \alpha$ is a base of the topology of the $\mathcal{T}_\alpha$-power $(C_2)^{UX}$ which is a $\mathcal{T}_\alpha$-coreflection of the $\mathcal{T}$-power $(C_2)^{UX}$. Since $[X,C_2]$ is a subspace of all continuous maps $X \longrightarrow C_2$ of the $\mathcal{T}$-power $(C_2)^{UX}$, $H_\alpha(X,C_2)$ is a subspace of all continuous maps $X \longrightarrow C_2$ of the $\mathcal{T}_\alpha$-power $(C_2)^{UX}$.

2.12. Proposition. Let α be an infinite cardinal, K a uniform filter space on $\alpha + 1$ and $C(\alpha) \square K = C(\alpha) \otimes K$. Then $H(K,C_2) = [K,C_2]$.

Proof. If X is a countable space, then $[X,C_2]$ is a first countable space so that $H_{\omega_0}(X,C_2) = [X,C_2]$ and therefore $H(X,C_2) = [X,C_2]$. Let α be a cardinal with $H(K,C_2) \neq [K,C_2]$. Denote by $\mathcal{U}_H$ the topology of the space $H(K,C_2)$, $\mathcal{U}$ the topology of $[K,C_2]$ and $\mathcal{B}$ the base of $\mathcal{U}$ for which $B \in \mathcal{B}$ if and only if $B = (\prod_{x\in X} V_x) \cap \mathcal{T}(K,C_2)$ where V_x are open subsets of C_2 and the set $\{x\in K: V_x = \{1\}\}$ is finite. The family $\mathcal{B}_\alpha = \{B \cap \mathcal{T}(K,C_2) : B \in \mathcal{B}^{\alpha}_K\}$ (see $\mathcal{B}^{\alpha}_X$ above) is a base of $H_\alpha(K,C_2)$. Let $V \in \mathcal{U}_H - \mathcal{U}$. Then there exists a collection $\mathcal{S} \subset \mathcal{B}_\alpha$ with $V = \bigcup_{B\in\mathcal{S}} B$. Put $\mathcal{S}_1 = \mathcal{S} \cap \mathcal{B}$ and $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$. Then there exists $B_0 \in \mathcal{S}_2$ with $B_0 \not\subset \bigcup_{B\in\mathcal{S}_1} B$ (otherwise $V \in \mathcal{U}$). For each $B \in \mathcal{S}$ put $E_B = \{x \in K : t(x) = 1$ for all $t \in B\}$ and $E_{B_0} = E$. Let $e \notin E$ and $p: K \longrightarrow E \cup \{e\}$ be the map given by $p(x) = x$ for each $x \in E$ and $p(x) = e$ otherwise. Let L denote the extremal quotient space (factor space) on $E \cup \{e\}$ corresponding to the map p. If $\alpha \notin E$, then $K - E$ is a neighbourhood of α so that L is a discrete space. If $\alpha \in E$, then the subset $\{\alpha, e\}$ is open and closed in L and the subspace P of L on the set $\{\alpha, e\}$ is isomorphic with C_2. Hence $L = P \sqcup D$ where D is a discrete space (on $E - \{\alpha\}$). The functor $H(-,C_2)$, $[-,C_2]: \mathcal{T}^{op} \longrightarrow \mathcal{T}$ preserves limits so that $H(P \sqcup D, C_2)$ is isomorphic with $H(P,C_2) \times H(D,C_2) = [P,C_2] \times \times [D,C_2]$ and this space is isomorphic with $[P \sqcup D, C_2]$ (P is countable and D discrete). Thus, $H(L,C_2) = [L,C_2]$. Now consider the map $H(p,1): H(L,C_2) \longrightarrow H(K,C_2)$ and put $W = H(p,1)^{-1}[V]$. Let $t \in W$ with $t(x) = 1$ for each $x \in E$ and $t(e) = 0$. If $B \in \mathcal{S}$ and $B \not\supset B_o$, then $E_B - E \neq \emptyset$ so that $H(p,1)(t) \notin B$. If $B \supset B_o$, then $B \in \mathcal{S}_2$. Let $\mathcal{O}$ be an arbitrary neighbourhood of t belonging to $\mathcal{B}$. Then there exists a finite set $I \subset E$ such that $\mathcal{O} = \{s \in H(L,C_2): s(x) = 1$ for each $x \in I\}$. The element $o \in \mathcal{O}$ for which $o(x) = 1$ for all $x \in I$ and $o(x) = 0$ otherwise does not belong to any $B \in \mathcal{S}$ with $B \supset B_o$. Thus $\mathcal{O}$ cannot be a subset of W so that W is not open in $H(L,C_2)$. But $H(p,1)$ is a continuous map - a contradiction.

2.13. Corollary. For any infinite cardinal α, $H(C(\alpha),C_2) = [C(\alpha),C_2]$.

2.14. Corollary. For any topological space X and any infinite cardinal α, $X \sqcap C(\alpha) = X \otimes C(\alpha)$.

Proof. From 2.13 it follows that $\mathcal{T}(X \sqcap C(\alpha),C_2) = \mathcal{T}(X \otimes C(\alpha),C_2)$.

2.15. Corollary. For any infinite cardinal α and any uniform filter space T on $\alpha + 1$ $H(T,C_2) = [T,C_2]$.

Proof. Immediate from 2.12, 2.14 and the symmetry of $\sqcap$.

2.16. Theorem. There exists (up to isomorphism) exactly one structure of closed category on the category $\mathcal{T}$.

Proof. Let X be a topological space and T a uniform filter space. Then by 2.15, $\mathcal{T}(X \square T, C_2) = \mathcal{T}(X \otimes T, C_2)$ and therefore $X \square T = X \otimes T$. Thus the tensor products $\square$ and $\otimes$ coincide on $\mathcal{L} \times \mathcal{L}$ and by 2.4 $\square = \otimes$.

2.17. Remark. Note that we have proved 2.16 without using the associativity of $\square$.

3. Closed structures on the category $\mathcal{T}_0$. The category $\mathcal{T}_0$ is an extremal epireflective subcategory of the category $\mathcal{T}$ (see e.g. [4],[5]). Therefore $\mathcal{T}_0$ is productive and monomorphism-closed (i.e. if $m: M \to X$ is a monomorphism and $X \in \mathcal{T}_0$, then $M \in \mathcal{T}_0$). Hence, if X,Y are T_0-spaces, then $X \otimes Y$ (see [6]) and $[X,Y]$ are T_0-spaces and it is easy to see that the restriction of $(\otimes, [-,-])$ to the category $\mathcal{T}_0$ is a closed structure on $\mathcal{T}_0$. This closed structure on $\mathcal{T}_0$ will be agaim (inaccurately) denoted by $(\otimes, [-,-])$.

The category $\mathcal{T}_0$ fulfils the conditions (1) - (3) of 2.1 so that without loss of generality we can suppose all closed structures on $\mathcal{T}_0$ to satisfy (a) - (f) of 2.1.

Similarly as in $\mathcal{T}$ we can show that in the category $\mathcal{T}_0$ the coreflective hull of the class $\mathcal{L}$ of all uniform ultraspaces is precisely $\mathcal{T}_0$. Hence, any tensor product on $\mathcal{T}_0$ is uniquely determined by its values on $\mathcal{L} \times \mathcal{L}$.

Recall that for any filter space $(A, a, \mathcal{F})$ the filter $\mathcal{F}$ is supposed to be free (i.e. $\bigcap \mathcal{F} = \emptyset$).

3.1. Proposition. Let $(\square, H)$ be a closed structure on $\mathcal{T}_0$ and α, β infinite cardinals. Let K, L be filter spaces on $\alpha + 1$, $\beta + 1$ respectively. Then $U(K \square L) = UK \times UL$ ($U: \mathcal{T}_0 \to$ Set is the forgetful functor).

Proof. Let $x \in \alpha$. Then $\{x\}$ is an open and closed subset of K so that $K = \{x\} \sqcup K'$. But then $K \square L = (\{x\} \sqcup K') \square L = (\{x\} \square L) \sqcup (K' \square L)$. Hence, $\{x\} \times (\beta + 1)$ is an open (and closed) subset of $K \square L$ for each $x \in \alpha$. Similarly, for each $y \in \beta$ $(\alpha + 1) \times \{y\}$ is an open subset of $K \square L$. Consequently, $P = ((\alpha + 1) \times (\beta + 1)) - \{(\alpha, \beta)\}$ is an open subset of $K \square L$ and $Q = (K \square L) - P$ is a closed subset of $K \square L$. Put $Q' = cl\{(\alpha, \beta)\}$. Clearly, $Q' \subset Q$. Define the maps $f: K \square L \to C_2$ by $f(t) = 0$ for each $t \in Q'$, $f(t) = 1$ otherwise and $g: K \square L \to C_2$ by $g(t) = 0$ for each $t \in Q$, $g(t) = 1$ for each $t \in P$. Then $f | UK \times UL = g | UK \times UL$ so that by 2.1(c) $f = g$. Therefore $Q = Q'$. Let $z \in Q - \{(\alpha, \beta)\}$ and $Q_z = cl\{z\}$. Since $K \square L$ is a T_0-space, $(\alpha, \beta) \notin Q_z$. The maps $f: K \square L \to C_2$; $f[Q_z] \subset \{0\}$, $f[(K \square L) - Q_z] = \{1\}$ and $g: K \square L \to C_2$; $g(t) = 1$ for each $t \in K \square L$ are continuous and $f | UK \times UL = g | UK \times UL$. Therefore $f = g$ and $Q = \{(\alpha, \beta)\}$.

Since any T_0-space X is an extremal quotient of a coproduct of a suitable family of filter spaces in the category $\mathcal{T}_0$, any extremal epimorphism in $\mathcal{T}_0$ is a surjection and any tensor product $\square$ on $\mathcal{T}_0$ preserves coproducts and extremal epimorphisms, we obtain:

3.2. <u>Proposition</u>. If $(\square, H)$ is a closed structure on $\mathcal{T}_0$ fulfilling the conditions (a) - (f) of 2.1, then it fulfils also (g) and (h).

Finally, one can easily see that 2.5 - 2.15 remain valid also for the category $\mathcal{T}_0$ (all spaces considered there are $\mathcal{T}_0$-spaces, $\mathcal{T}_{0\alpha} = \mathcal{T}_\alpha \cap \mathcal{T}_0$ and for any T_0-space X the $\mathcal{T}_{0\alpha}$-coreflection of X coincides with the $\mathcal{T}_\alpha$-coreflection of X).

Thus, we can state:

3.3. Theorem. There exists (up to isomorphism) exactly one structure of closed category on the category $\mathcal{T}_0$.

References

[1] ČINČURA J.: Tensor products in initially structured categories, Math. Slovaca, to appear.

[2] COMFORT W.W. - NEGREPONTIS S.: The Theory of Ultrafilters, Springer, Berlin 1974.

[3] EILENBERG S. - KELLY G.M.: Closed categories, Proc. of the Conf. on Categorial Algebra, La Jolla 1965. Springer, New York 1966, 421-562.

[4] HERRLICH H.: Topologische Reflexionen und Coreflexionen, Lecture Notes in Math., Vol. 78, Springer, Berlin 1968.

[5] HERRLICH H. - STRECKER G.E.: Category Theory, Boston 1973.

[6] KNIGHT C.J. - MORAN W. - PYM J.S.: The topologies of separate continuity, I. Proc. Camb. Phil. Soc. 68 (1970), 663-671.

[7] MAC LANE S.: Categories for the Working Mathematician, Springer, New York 1971.

[8] NIEDERLE H.: A remark on tensor products in concrete categories with constant mappings (Czech), Arch. Math. (Brno), to appear.

[9] PAVELKA J.: Tensor products in the category of convergence spaces, Comment. Math. Univ. Carolinae 13(1972), 693-709.

[10] WISCHNEWSKY M.B.: Aspects of universal algebra in initialstructure categories, Cahier Topo. Geo. Diff. XV, 1974, 1-27.

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