Jan Malý

A note on separation of sets by approximately continuous functions

Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 2, 207--212

Persistent URL: `http://dml.cz/dmlcz/105922`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20,2 (1979)

A NOTE ON SEPARATION OF SETS BY APPROXIMATELY CONTINUOUS FUNCTIONS

Jan MALÝ

Abstract: An example of two G_δ-sets with disjoint closures in density topology, which cannot be separated by any approximately continuous function is given.

Key words: Density topology, d-derivative of a set, separation properties of approximately continuous functions.

AMS: Primary 26A15
Secondary 54D15

According to Z. Zahorski [4] given any G_δ d-closed (i.e. closed in the density topology) set $A \subset \mathbf{R}$ there exists a bounded approximately continuous function f such that $A = \{x : f(x) = 0\}$. Consequently, for every pair A, B of disjoint G_δ d-closed sets there is an approximately continuous function f, which separates A and B in the sense that $0 \le f \le 1$, $f = 0$ on A, $f = 1$ on B.

The last assertion is not generally true, if we suppose A, B to be G_δ sets with disjoint d-closures only, as follows from the example, given in this paper. This answers negatively to the problem posed by M. Laczkovich [2].

Denote by λ the Lebesgue measure and by λ^* the corresponding outer measure. If $E \subset \mathbf{R}$ is an arbitrary set and $x \in \mathbb{R}$, then we define the outer density of A at x by

$$D(E,x) = \overline{\lim} \frac{\lambda^*(\langle x-h,\ x+h \rangle \cap E)}{2h}$$

and the inner density by $d(E,x) = 1 - D(\mathbb{R} - E,x)$. The collection of all sets having the inner density one in each its point forms topology, which will be called the density topology (d-topology). It is easy to see that the d-derivative of a set $E \subset \mathbb{R}$ will be the set $\mathcal{D} E = \{x \in \mathbb{R} : D(E,x) > 0\}$ and the d-closure of E will be $E \cup \mathcal{D} E$.

<u>Lemma 1</u>. For an arbitrary bounded interval $I = (a,b)$ and $c \in (0,1)$ there is an open set $G(I,c) \subset I$ with the following properties:

(1) $\{a,b\} \subset \mathcal{D} G(I,c)$.

(2) If $x \in \mathbb{R} - I$ and $h > 0$, then
$\lambda(G(I,c) \cap \langle x-h,\ x+h \rangle) \le 2ch$.

<u>Proof</u>. Put

$$d_n = c((n+1)^{-1} - (n+2)^{-1}),$$

$$L = [\bigcup_{n=1}^{\infty} (a+n^{-1} - d_n,\ a + n^{-1})] \cap (a, a + \tfrac{1}{2} c(b-a)),$$

$$R = [\bigcup_{n=1}^{\infty} (b-n^{-1},\ b - n^{-1} + d_n)] \cap (b - \tfrac{1}{2} c(b-a), b),$$

$$G(I,c) = L \cup R.$$

The property (1) is evidently satisfied, concretely

$$D(G(I,c),a) = D(G(I,c),b) = \tfrac{1}{2} c$$

(choose $h = n^{-1}$, $n = 1,2,\ldots$). We shall prove (2) for $x \le a$.

We claim

(3) $\lambda(\langle x-h,\ x+h \rangle \cap L) \le ch$.

Indeed, consider $m \in \mathbb{N}$, $(m+1)^{-1} \leq h \leq m^{-1}$. Then

$$\lambda(\langle x-h, x+h\rangle \cap L) \leq \lambda(\langle a-h, a+h\rangle \cap L) \leq$$
$$\leq \lambda(\langle a-m^{-1}, a+m^{-1}\rangle \cap L) = c(m+1)^{-1} \leq ch.$$

On the other hand,

$$\lambda(\langle x-h, x+h\rangle \cap R) \leq ch. \tag{4}$$

If $h < \frac{1}{2}(b-a)$, then (4) holds trivially, since

$$\langle x-h, x+h\rangle \cap R = \emptyset.$$

If $h \geq \frac{1}{2}(b-a)$, then

$$\lambda(\langle x-h, x+h\rangle \cap R \leq \lambda R \leq \frac{1}{2} c(b-a) \leq ch.$$

From (3) and (4) we immediately obtain (2).

Denote by C the Cantor's discontinuum (or an arbitrary perfect nowhere dense set with $\lambda C = 0$, $\inf C = 0$, $\sup C = 1$). There are open disjoint intervals (a_i, b_i) $(i = 1,2,\dots)$ such that

$$C = \langle 0,1\rangle - \bigcup_{i=1}^{\infty} (a_i, b_i).$$

The set $\bigcup_{i=1}^{\infty} \{a_i, b_i\}$ will be denoted by S. Further put $B = C - S$. Finally, consider

$$A = \bigcup_{i=1}^{\infty} G((a_i, b_i), 2^{-i}).$$

The sets A, B and S have the following important properties:

<u>Lemma 2</u>. (i) $\mathcal{D} A \cap B = \emptyset$.

(ii) $S \subset \mathcal{D} A$.

(iii) S is not a G_δ.

(iv) A and B are G_δ sets with disjoint d-closures.

<u>Proof</u>. (i) Let $x \in B$. Choose $\varepsilon > 0$. Find a positive integer k with $2^{-k} < \varepsilon$. There is $\delta > 0$ such that

$$\langle x-\delta, x+\delta\rangle \cap \bigcup_{i=1}^{k}(a_i,b_i) = \emptyset .$$

For every $i > k+1$ and h, $0 < h < \delta$ we have

$$\lambda(\langle x-h,\ x+h\rangle \cap G((a_i,b_i),\ 2^{-i}) \leq 2^{-i+1}h .$$

Thus

$$\lambda(\langle x-h,\ x+h\rangle \cap A) \leq \sum_{i=k+1}^{\infty} 2^{-i+1}h < 2\varepsilon\ h .$$

Since $\varepsilon > 0$ may be chosen arbitrary, $x \notin \mathcal{D} A$.

(ii) For every $i = 1,2,\dots$ we obtain from (1)

$$\{a_i,b_i\} \subset \mathcal{D}\, G((a_i,b_i),2^{-i}) \subset \mathcal{D} A .$$

(iii) The set S is of the first category and dense in the Baire space C, and thus it is not a G_δ.

(iv) Obviously, A is open and $\langle 0,1\rangle - B = \bigcup_{i=1}^{\infty} \langle a_i,b_i\rangle$. Since $\lambda B = 0$, we have $\mathcal{D} B = \emptyset$. Clearly, $A \cap B = \emptyset$ and using (i) we obtain

$$c\ell_d A \cap c\ell_d B = c\ell_d A \cap B = \mathcal{D} A \cap B = \emptyset .$$

We shall show that there exists a set whose d-derivative is not a G_δ:

<u>Theorem 1.</u> If A is as above, then $\mathcal{D} A$ is not a G_δ.

<u>Proof.</u> It is an easy consequence of Lemma 2, parts (i), (ii) and (iii).

<u>Definition.</u> A function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is said to be approximately continuous if for every $x \in \mathbb{R}$ there is a set M such that $x \in M$, $d(x,M) = 1$ and $f|_M$ is continuous at x.

The approximately continuous functions are just the continuous mappings from the density topology to the euclidean one. Thus,

(5) $\{x:f(x) = 0\}$ is d-closed

for any approximately continuous function f.
Since any approximately continuous function f is of the Baire class one (see for example [3]), it follows that

(6) $\{x:f(x) = 0\}$ is a G_δ.

Theorem 2. Assume that an approximately continuous function f vanishes on A. Then there exists $x \in B$ with $f(x) = 0$. (A, B are as above.)

Proof. Denote $M = \{x:f(x) = 0\}$. By (5), $\mathcal{D} A \subset M$ and by (6), M is a G_δ. Thus $\mathcal{D} A \cap C \neq M \cap C$ according to Theorem 1. Hence there is a point $x \in M \cap C - \mathcal{D} A \subset C - S = B$.

Corollary. The sets A and B cannot be separated by any approximately continuous function.

Remark. It is not difficult to prove that the d-derivative of any set is always a $G_{\delta\sigma}$. We have seen that the d-derivative need not be a G_δ. On the other hand, it need not be a F_σ as well. Indeed, let M be a measurable set such that $\lambda(I \cap M) > 0$ and $\lambda(I - M) > 0$ for every interval I. Then either M or $\mathbb{R} - M$ is not a F_σ. Let us remark only, that if M is a set whose d-derivative is not a F_σ, then the upper symmetric derivative of the function $x \longmapsto \lambda(\langle 0,x\rangle \cap M)$ is not of the first class of Baire (although the upper derivative, or even the upper symmetric derivative of arbitrary function is of the second class of Baire, see e.g. [1]).

R e f e r e n c e s

[1] O. HÁJEK: Note sur la mesurabilité B de la dérivée supérieure, Fund. Math. 44(1957), 238-240.

[2] M. LACZKOVICH: Separation properties of some subclasses of Baire 1 functions, Acta Math. Acad. Sci. Hungar 26(1975), 405-412.

[3] J. LUKEŠ and L. ZAJÍČEK: The insertion of G_δ sets and fine topologies, Comment. Math. Univ. Carolinae 18(1977), 101-104.

[4] Z. ZAHORSKI: Sur la première dérivée, Trans. Amer. Math. Soc. 69(1950), 1-54.

Matematicko-fyzikální fakulta

Universita Karlova

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 18.12. 1978)