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FINITE ELEMENT ANALYSIS OF THE SIGNORINI PROBLEM

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Abstract: Finite element analysis of the Signorini problem is given. The paper extends results, contained in [2], [10], where the analysis for polygonal domains is studied, only. Finite dimensional approximations K_h of the closed convex set K of admissible displacements are external, in general, i.e. $K_h \subset K$, for the Signorini problem over domains with curved boundary. The main difficulty arises in semi-coercive cases, where the coerciveness of the functional of total potential energy on K doesn't ensure the same property on $\bigcup_h K_h$. The rate of convergence is studied, provided the exact solution is smooth enough. Since the regularity assumptions are not satisfied, in general, we prove the convergence of u_h to u, without any regularity hypothesis. These results can be extended to contact problems of elastic bodies, see [5].

Key words: Finite elements, numerical solution of variational inequalities.

AMS: 65N30

Notations. Let $\Omega \subset R_2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. $H^k(\Omega)$ ($k \geq 0$ integer) denotes the usual Sobolev space of functions, derivatives of which up to the order k are square integrable in Ω. We write $H^0(\Omega) = L^2(\Omega)$, where the scalar product will be denoted by (,). We set $\mathcal{H}^k = H^k(\Omega) x H^k(\Omega)$. The norm in $\mathcal{H}^k$, introduced in the usual way, will be denoted by $\| \ \|_{k,\Omega}$ or simply $\| \ \|_k$. In the next, the summation convention will be used: a repeated index implies always the summation over the range 1, 2. Instead of $\partial v_i/\partial x_j$, we shall write $v_{i,j}$.

1. Setting of the problem. Let an elastic body occupy the bounded domain $\Omega \subset R_2$, Lipschitz boundary of which is decomposed as follows:

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_\tau \cup \bar{\Gamma}_S \cup \bar{\Gamma}_0 ,$$

where $\Gamma_u, \Gamma_\tau, \Gamma_S$ and Γ_0 are mutually disjoint parts open in $\partial\Omega$ and $\Gamma_S \neq \emptyset$. Let $F = (F_1, F_2) \in \mathcal{H}^0$ and $P = (P_1, P_2) \in (L^2(\Gamma_\tau))^2$ are prescribed body forces and surface loads, respectively. The displacement field $u = (u_1, u_2)$ is a classical solution of the so called Signorini problem, if

$$u = 0 \text{ on } \Gamma_u ,$$

$$u_n = u_i n_i = 0,\ T_t = \tau_{ij} n_j t_i = 0 \text{ on } \Gamma_0 ,$$

$$\tau_{ij} n_j = P_i,\ i = 1,2 \text{ on } \Gamma_\tau ,$$

$$u_n \leq 0,\ T_n = \tau_{ij} n_j n_i \leq 0,\ u_n T_n = 0 \text{ on } \Gamma_S$$

and the equilibrius equations

$$\tau_{ij,j} + F_i = 0,\ i = 1,2$$

hold in Ω . Here u_n denotes the normal component of the displacement vector u. $n = (n_1, n_2)$ and $t = (t_1, t_2) = (-n_2, n_1)$ are the outward unit normal and the tangential vector to $\partial\Omega$. Similarly T_n and T_t are normal and tangential components, respectively, of the stress vector $T = (T_1, T_2) = (\tau_{1j} n_j, \tau_{2j} n_j)$. The stress tensor $\tau = (\tau_{ij})^2_{i,j=1}$ and the strain tensor $\varepsilon = (\varepsilon_{ij})^2_{i,j=1}$ are related by means of the generalized Hooke's law

$$\tau_{ij} = \tau_{ij}(u) = c_{ijkl}\, \varepsilon_{kl}(u),$$

where $\varepsilon_{kl} = \varepsilon_{kl}(u) = 1/2(u_{k,l} + u_{l,k})$. The elastic coefficients $c_{ijkl} \in L^\infty(\Omega)$ satisfy the symmetry conditions:

$$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{a.e. in } \Omega$$

and the condition of ellipticity:

(1.1) $\exists\, \alpha_0 = \text{const.} > 0 : c_{ijkl} e_{ij} e_{kl} \geq \alpha_0 e_{ij} e_{ij}$

holds for any symmetric e_{ij}.

In order to define the variational solution, we introduce the space of virtual displacements

$$V = \{ v \in \mathcal{H}^1 \mid v = 0 \text{ on } \Gamma_u, v_n = 0 \text{ on } \Gamma_0 \}$$

and the closed convex set of admissible displacements

$$K = \{ v \in V \mid v_n \leq 0 \text{ on } \Gamma_S \} .$$

Let

$$\mathcal{L}(v) = 1/2A(v,v) - L(v),$$

where

$$A(u,v) = (\tau_{ij}(u), \varepsilon_{ij}(v)), \quad L(v) = (F_i, v_i) + \int_{\Gamma_\tau} P_i v_i \, ds,$$

be the functional of the total potential energy. An element $u \in K$ will be called a <u>weak solution</u> of the Signorini problem, if

($\mathcal{P}$) $\mathcal{L}(u) \leq \mathcal{L}(v) \quad \forall\, v \in K.$

The classical and variational formulations are equivalent in some sense. If $\Gamma_u \neq \emptyset$ (coercive case) then there exists a unique solution of ($\mathcal{P}$) (see [1]). If $\Gamma_u = \emptyset$ (semi-coercive case), some sufficient conditions for the existence and uniqueness of the solution of ($\mathcal{P}$) can be formulated (for details see [1],[2]).

2. <u>Approximation of</u> ($\mathcal{P}$). In this Section we describe the construction of finite-dimensional approximations of K. For the sake of simplicity we restrict ourselves to the case, when only Γ_S is curved. Let Υ be a continuous concave (it

is not necessary) function defined on $\langle a,b \rangle$, the graph of which is Γ_S . We choose (m + 1) points $A_1,\dots,A_{m+1}$ on Γ_S in such a way that A_1, A_{m+1} are boundary points of Γ_S . Let A_i, $A_{i+1} \in \Gamma_S$, $Q \in \Omega$. By a curved element T we call a closed set bounded by the straight-lines QA_i, QA_{i+1} and the arc $\widehat{A_iA_{i+1}}$. The minimal interior angle of the curved element T is called the minimal angle of the curved element T. A triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ contains curved elements along Γ_S and internal triangular elements. By the symbols h and ϑ we denote the maximal diameter and the minimal interior angle, respectively, of all elements $T \in \mathcal{T}_h$. We shall assume only the so called regular systems of triangulations:

a constant $\vartheta_0 > 0$ exists, independent of h and such that

$$\vartheta \geq \vartheta_0 \text{ if } h \to 0+ .$$

A family of triangulations will be called $\alpha - \beta$ regular, if $\vartheta_0 = \alpha$ and

$$\frac{h}{h_{min}} \leq \beta ,$$

where h_{min} is the minimal diameter of all $T \in \mathcal{T}_h$. Define

$$V_h = \{ v \in (C(\bar{\Omega}))^2 \cap V \mid v|_T \in (P_1(T))^2, \quad \forall T \in \mathcal{T}_h \}$$

and

$$K_h = \{ v \in V_h \mid v.n(A_i) \leq 0, \; i = 1,\dots,m + 1 \} ,$$

where $P_1(T)$ denotes the set of linear polynomials, defined on T. It is easy to see that K_h represents a finite-dimensional approximation of K and $K_h \not\subset K$, in general.

An approximation of the Signorini problem is defined as the solution of the following problem:

$$(\mathcal{P}_h) \qquad \begin{cases} \text{find } u_h \in K_h \text{ such that} \\ \mathcal{L}(u_h) \le \mathcal{L}(v) \quad \forall\, v \in K_h. \end{cases}$$

3. Error estimates. In this Section we establish the rate of convergence of u_h to u, provided the both problems $(\mathcal{P})$ and $(\mathcal{P}_h)$ have solutions and u is smooth enough. First let us recall some well-known results, needed in what follows.

Lemma 3.1. It holds

$$(3.1)\quad 1/2\, A(u - u_h, u - u_h) \le \{ L(u - v_h) + L(u_h - v) + 1/2\, A(v_h - u, v_h - u) + A(u, v - u_h) + A(u, v_h - u)\}$$
$$\forall\, v \in K,\ v_h \in K_h.$$

Proof. See [3].

Theorem 3.1. Let us assume

$$(3.2)\quad \forall\, v \in K \quad \exists\, v_h \in K_h:\ \| v - v_h \|_1 \longrightarrow 0,\ h \longrightarrow 0+\,;$$

$$(3.3)\quad v_h \in K_h,\ v_h \rightharpoonup v \text{ (weakly) in } \mathcal{H}^1 \text{ implies } v \in K.$$

Let there exist $\gamma_0 > 0$ such that

$$A(v,v) \ge \gamma_0 \| v \|_1^2$$

holds for any $v \in V$. Then

$$\| u - u_h \|_1 \longrightarrow 0,\ h \longrightarrow 0+ .$$

Proof. See [4].

Theorem 3.2. Let us suppose that $\mathcal{L}$ is coercive on $\bigcup_{h>0} K_h$, i.e.

$$(3.4)\quad v_h \in K_h,\ \| v_h \|_1 \longrightarrow +\infty \quad \text{implies} \quad \mathcal{L}(v_h) \longrightarrow +\infty$$

and let (3.2) and (3.3) be satisfied. Let there exist $\gamma_0 > 0$

such that

$$A(v,v) \geq \gamma_0 \, |v|^2 ,$$

where $|v| = (\varepsilon_{ij}(v), \varepsilon_{ij}(v))^{1/2}$. Then

$$|u - u_h| \longrightarrow 0, \; h \longrightarrow 0+ .$$

Moreover if the solution u of $(\mathcal{P})$ is unique, then

$$\| u - u_h \|_1 \longrightarrow 0, \; h \longrightarrow 0+ .$$

Proof. See [5].

Lemma 3.2. Let $\Omega \subset R_2$ be a bounded convex domain, the boundary of which is twice continuously differentiable and let $\{\mathcal{T}_h\}$ be a $\alpha - \beta$ regular system of triangulations with $\alpha < \pi/8$, $\beta = 2$. Then

$$\| u - u_I \|_{0,\partial\Omega} \leq ch^{3/2} \| u \|_{2,\Omega} \qquad \forall u \in H^2(\Omega),$$

where u_I denotes the piecewise-linear Lagrange interpolate of u, $c > 0$ is independent of $h > 0$.

Proof. See [6].

Lemma 3.3. Let $v \in P_1(T)$, where T is a closed triangular element. Let T_h be the triangle generated by replacing the curved side by its chord. Then

$$\| v \|^2_{1,\Delta(T,T_h)} \leq ch \, \| v \|^2_{1,T_h} ,$$

where $\Delta(T,T_h) = (T \setminus T_h) \cup (T_h \setminus T)$ and $c > 0$ is independent of h.

Proof. See [7].

Now we recall the well-known Green's formula. To this end we define

$$S(\Omega) = \{\tau \in (L^2(\Omega))^4 \mid \tau_{ij} = \tau_{ji} \text{ a.e. in } \Omega\}$$

$$Y(\Omega) = \{\tau \in S(\Omega) \mid \tau_{ij,j} \in L^2(\Omega),\ i = 1,2\},$$

where $\tau_{ij,j}$ is taken in the sense of distributions. Then there exists a unique $T \in L(Y(\Omega), (H^{-1/2}(\partial\Omega))^2)$ such that

$$(\tau_{ij}, \varepsilon_{ij}(v)) = -(\tau_{ij,j}, v_i) + \langle T(\tau), v \rangle$$

holds for any $\tau \in Y(\Omega)$ and $v \in \mathscr{H}^1$. $\langle\ ,\ \rangle$ denotes the duality between $(H^{-1/2}(\partial\Omega))^2$ and $(H^{1/2}(\partial\Omega))^2$. Henceforth we assume for simplicity that $T(\tau(u)) \in (L^2(\partial\Omega))^2$, so that

$$T(\tau) = (\tau_{1j}(u)n_j,\ \tau_{2j}(u)n_j)$$

and

$$\langle T(\tau), v\rangle = \int_{\partial\Omega} T_i v_i \, ds.$$

<u>Theorem 3.3</u>. Let both problems $(\mathcal{P})$ and $(\mathcal{P}_h)$ have solutions u and u_h, respectively. Let $u \in K \cap \mathscr{H}^2$, $\tau(u) \in Y(\Omega)$ and $T_n(u) \in L^2(\Gamma_S)$. Let the system of triangulations $\{\mathcal{T}_h\}$ satisfy the assumptions of Lemma 3.2 and Ψ, describing Γ_S, be from $C^3(\langle a,b\rangle)$. If the norms $\|u_h\|_1$ remain bounded then

$$|u - u_h| \leq c(u)h^{3/4}.$$

<u>Proof</u>. Using the definition of $(\mathcal{P})$, Green's formula and (3.1), we deduce

$$1/2\, A(u - u_h, u - u_h) \leq 1/2\, A(v_h - u, v_h - u) +$$

$$+ \int_{\Gamma_S} T_n(v_n - u_{hn})\, ds + \int_{\Gamma_S} T_n(v_{hn} - u_n)\, ds \quad \forall v \in K, v_h \in K_h.$$

Let $v_h = u_I$, where u_I is the piecewise linear Lagrange interpolate of u on Ω. It is easy to see that $u_I \in K_h$ and

$$
(3.5) \qquad \begin{cases} A(u_I - u, u_I - u) \leq ch^2 \| u \|^2_{2,\Omega} \\ \int_{\Gamma_S} T_n(u_I - u).n \, ds \leq c \| u_I - u \|_{o,\Gamma_S} \leq \\ \qquad \leq ch^{3/2} \| u \|_{2,\Omega} \end{cases}
$$

where the assertion of Lemma 3.2 has been used. The most difficult is to estimate the term

$$
(3.6) \qquad \int_{\Gamma_S} T_n(v_n - u_{hn}) \, ds.
$$

In what follows we shall construct a function $v \in K$ such that (3.6) is small. We identify the origin of coordinate system (x_1, x_2) with the point A_i. Let Σ_i be a closed set brunded with the arc $\widehat{A_i A_{i+1}} \equiv s_i \subset \Gamma_S$ and the chord $A_i A_{i+1}$. Let $x \in \in \Sigma_i$. By the symbol $P(x)$ and $Q(x)$, respectively, we denote the intersection of the perpendicular line through the point x with s_i and $A_i A_{i+1}$, respectively. Let us define functions U_h, $\tilde{U}_h$ on $\bigcup_{i=1}^{m} \Sigma_i$ by means of the following relations:

$$
U_h(x) = u_h(x).n(P(x)),
$$

$$
\tilde{U}_h(x) = u_h(Q(x)).n(P(x)) = \tilde{u}_h(x).n(P(x)),
$$

where we set $\tilde{u}_h(x) = u_h(Q(x))$. Clearly

$$
U_h(x) = \tilde{U}_h(x), \quad x \in A_i A_{i+1}.
$$

Let $\Phi_i(x)$, $x \in A_i A_{i+1}$ be the linear Lagrange interpolate of U_h on $A_i A_{i+1}$ and let us define $\tilde{\Phi}$ on $\bigcup_{i=1}^{m} \Sigma_i$ as follows:

$$
\tilde{\Phi}(x) = \Phi_i(Q(x)), \; x \in \Sigma_i, \; i = 1, \dots, m+1.
$$

It is readily seen that $\tilde{\Phi} \leq 0$ on Γ_S. We shall estimate

$$
\| \tilde{\Phi} - U_h \|_{o,\Gamma_S} .
$$

We may write:

(3.7) $\| \tilde{\Phi} - u_h \|_{o,\Gamma_S} \le \| \tilde{\Phi} - \tilde{U}_h \|_{o,\Gamma_S} + \| \tilde{U}_h - U_h \|_{o,\Gamma_S}$,

$$\| \tilde{U}_h - U_h \|^2_{o,\Gamma_S} = \sum_{i=1}^{m} \| \tilde{U}_h - U_h \|^2_{o,s_i} \le 2 \sum_{i=1}^{m} \| u_h - \tilde{u}_h \|^2_{o,s_i} .$$

Let q be the arc's parameter of the point $P(x) = (P_1(x), P_2(x))$ and denote $Q_1(x) = x_1$. Then for $j = 1,2$ we have

$$u_{hj}(q) - \tilde{u}_{hj}(q) = \int_0^{P_1(x)} \frac{\partial}{\partial x_2} (u_{hj} - \tilde{u}_{hj})\, dx_2 = $$

$$= \int_0^{P_1(x)} \frac{\partial}{\partial x_2} u_{hj}(x_1, x_2)\, dx_2 .$$

Integrating and using Fubini's theorem we obtain

$$\| u_{hj} - \tilde{u}_{hj} \|^2_{o,s_i} \le ch^2 | u_h |^2_{1,\Sigma_i} \qquad j = 1,2.$$

From this and Lemma 3,3 we have

(3.8) $\| U_h - \tilde{U}_h \|^2_{o,\Gamma_S} \le ch^2 \sum_{i=1}^{m} | u_h |^2_{1,\Sigma_i} \le ch^3 | u_h |^2_{1,\Omega}$

Let us estimate $\| \tilde{\Phi} - \tilde{U}_h \|^2_{o,\Gamma_S}$.

$$\| \tilde{\Phi} - \tilde{U}_h \|^2_{o,\Gamma_S} = \sum_{i=1}^{m} \| \tilde{\Phi} - \tilde{U}_h \|^2_{o,s_i} .$$

$$\tilde{\Phi}(q) - \tilde{U}_h(q) = \int_0^{Q_1(x)} \frac{d}{dx_1} (\Phi_i(x_1,0) - \tilde{U}_h(x_1,0))\, dx_1 + $$

$$\int_0^{P_2(x)} \frac{d}{dx_2} (\Phi_i(Q_1(x),x_2) - \tilde{U}_h(Q_1(x),x_2)\, dx_2 = $$

$$\int_0^{Q_1(x)} \frac{d}{dx_1} (\Phi_i(x_1,0) - \tilde{U}_h(x_1,0))\, dx_1 .$$

Since $\Psi \in C^3(\langle a,b \rangle)$, we have $\tilde{U}_h \in H^2(A_i A_{i+1})$. Hence

$$(3.9) \qquad |\tilde{\Phi}(q) - \tilde{U}_h(q)|^2 \leq ch\ |\Phi_i - \tilde{U}_h|^2_{1,A_iA_{i+1}} \leq$$

$$\leq ch^3 |\tilde{U}_h|^2_{2,A_iA_{i+1}} .$$

As $\tilde{U}_h(x) = \tilde{u}_h(x).n(P(x))$ and $\tilde{u}_h \in P_1(A_iA_{i+1})$, we may write

$$|\tilde{U}_h|^2_{2,A_iA_{i+1}} \leq c\ |u_h|^2_{1,A_iA_{i+1}} .$$

Thus, (3.9) and the inverse inequality between $H^1(a_iA_{i+1})$ and $H^{1/2}(A_iA_{i+1})$ yield:

$$(3.10) \qquad \|\tilde{\Phi} - \tilde{U}_h\|^2_{o,s_i} \leq ch^4 \|u_h\|^2_{1,A_iA_{i+1}} \leq$$

$$\leq ch^3 \|u_h\|^2_{1/2,A_iA_{i+1}} .$$

Adding (3.10) for $i = 1,\dots,m$ we obtain:

$$(3.11) \qquad \|\tilde{\Phi} - \tilde{U}_h\|^2_{o,\Gamma_S} \leq ch^3 \|u_h\|^2_{1/2,\Gamma_h} ,$$

where $\Gamma_h = \bigcup_{i=1}^{m} A_iA_{i+1}$ is the polygonal approximation of Γ_S. Using the trace's theorem (see [8]) we obtain:

$$\|u_h\|^2_{1/2,\Gamma_h} \leq c\ \|u_h\|^2_{1,\Omega\setminus\bigcup_i \Sigma_i} \leq c\ \|u_h\|^2_{1,\Omega} ,$$

where $c > 0$ doesn't depend on h for h sufficiently small. Using these estimates, (3.7),(3.8) and (3.11) we deduce

$$(3.12) \qquad \|\tilde{\Phi} - U_h\|_{o,\Gamma_S} \leq ch^{3/2} \|u_h\|_{1,\Omega} .$$

Next, let $v \in V$ be such that

$$v.n = \tilde{\Phi} \quad \text{on } \Gamma_S .$$

Then $v.n. \leq 0$ on Γ_S , consequently $v \in K$. Finally we may write

$$(3.13)\qquad \int_{\Gamma_S} T_n(v_n - u_{hn})\, ds = \int_{\Gamma_S} T_n(\tilde{\Phi} - U_h)\, ds \leq$$
$$\leq ch^{3/2} \| u_h \|_{1,\Omega} .$$

Since the norms $\| u_h \|_{1,\Omega}$ remain bounded, the assertion of Theorem now follows from (3.5),(3.13) and (1.1).

Remark 3.1. Coercive case is very simple. Both problems, ($\mathcal{P}$) and ($\mathcal{P}_h$) have only one solution u and u_h, respectively. Using the Korn´s inequality (see [1]) we obtain the rate of convergence in $\mathcal{H}^1$ - norm, i.e.:

$$\| u - u_h \|_1 = O(h^{3/4}).$$

Moreover, the norms $\| u_h \|_1$ are bounded. More difficult are the semi-coercive cases. One of the first questions is, if $\mathcal{L}$ is coercive on $\bigcup_h K_h$. As $K_h \not\subset K$, in general, the coerciveness of $\mathcal{L}$ on $\bigcup_h K_h$ doesn´t follow from the same property on K. Coerciveness, together with (3.2) imply boundedness of $\| u_h \|_1$. That is why we had to assume the boundedness of $\| u_h \|_1$ explicitly. However, in some special cases, we can prove (3.4). Here, we present one of the possible situations.

Let

$$\mathcal{R} = \{ \rho = (\rho_1, \rho_2), \rho_1 = a_1 - bx_2, \ \rho_2 = a_2 + bx_1, \ a_1, a_2, b \in R_1 \}$$

be the space of rigid body displacement,

$$\mathcal{R}^* = \{ \rho \in R \cap K \mid \rho \in \mathcal{R}^* \Longrightarrow -\rho \in \mathcal{R}^* \},$$
$$\mathcal{R}_V = \mathcal{R} \cap V.$$

Assume that

$$(3.14)\qquad \mathcal{R}^* = \{0\}, \ \dim \mathcal{R}_V = 1,$$

(3.15) $L(\varphi) \neq 0 \ \forall \varphi \in \mathcal{R}_V - \{0\}$,

(3.16) $K \cap \mathcal{R} = \{0\}$.

Then

$$|u|^2 + \beta(u) \geq c \|u\|_1^2 \qquad \forall u \in V,$$

where $\beta(u) = \int_{\Gamma_S} (u_n^+)^2 \, ds$ (see [2],[9]).

Let $v_h \in K_h$, $\|v_h\|_1 \longrightarrow +\infty$. Then

(3.17) $\mathcal{L}(v_h) = 1/2A(v_h, v_h) - L(v_h) + \beta(v_h) - \beta(v_h) \geq$

$$\geq c \|v_h\|_1^2 - \beta(v_h) - c_1, \quad c, c_1 > 0.$$

Let $\tilde{\Phi}_h$ be a non-positive function on Γ_S such that

$$\|v_{hn} - \tilde{\Phi}_h\|_{0,\Gamma_S} \leq ch^{3/2} \|v_h\|_{1,\Omega} .$$

The construction of such a function is given in the proof of Theorem 3.3. Then

$$\beta(v_h) = \int_{\Gamma_S} (v_{hn}^+)^2 \, ds \leq \int_{\Gamma_S} ((v_{hn} - \tilde{\Phi}_h)^+)^2 \, ds \leq ch^3 \|v_h\|_{1,\Omega}^2 .$$

From this and (3.17)

(3.18) $\mathcal{L}(v_h) \geq c(1 - h^3) \|v_h\|_1^2 \longrightarrow +\infty$ if $\|v_h\|_1 \longrightarrow +\infty$.

Now, combining (3.18) with (3.2) we obtain the boundedness of the sequence $\|u_h\|_1$. Moreover, (3.15) ensures the uniqueness of the solution u and u_h.

The sufficient conditions, when (3.2) holds, are given in

<u>Lemma 3.5</u>. Let us suppose that $\Gamma_u \cap \Gamma_S = \emptyset$, $\bar{\Gamma}_S \cap \bar{\Gamma}_o = \emptyset$ and there exists only a finite number of boundary points $\bar{\Gamma}_\tau \cap \bar{\Gamma}_S$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_\tau \cap \bar{\Gamma}_o$. Then the set

$$\mathfrak{M} = K \cap (C^\infty(\bar{\Omega}))^2$$

is dense in K in $\mathcal{H}^1$-norm.

Proof. The proof for polygonal domains is given in [10], but its slight modification gives the same density result also in our case.

In the above error estimates we needed strong regularity assumptions, concerning the solution u. Unfortunately, there are no reasons to expect such a great smoothness. This is why we are going to prove the convergence of u_h to u without estimating the rate of convergence, using no regularity assumptions. According to Theorems 3.1 and 3.2, it remains to analyse the condition (3.3).

Lemma 3.6. The condition (3.3) holds.

Proof. Let $v_h \in K_h$ be such that

$$v_h \rightharpoonup v \text{ in } V, \quad h \to 0+ . \tag{3.19}$$

It is sufficient to show that $v.n \leq 0$ on Γ_S or equivalently

$$\int_{\Gamma_S} v.n.\varphi \, ds \leq 0$$

for any $\varphi \in C^1(\langle a,b \rangle)$, $\varphi \geq 0$ on $\langle a,b \rangle$.
Since the trace mapping is completely continuous from V into $(L^2(\Gamma_S))^2$, we have

$$v_h \to v \text{ in } (L^2(\Gamma_S))^2, \; h \to 0+ \tag{3.20}$$

hence

$$v_{hn} \to v_n \text{ in } L^2(\Gamma_S), \quad h \to 0+ .$$

Let Ψ_h be the piecewise linear function defined on $\langle a,b \rangle$, nodes of which are the points $A_1, \dots, A_{m+1}$. Then

$$\Gamma_{Sh} = \{(x_1,x_2), x_1 \in \langle a,b \rangle, \; x_2 = \Psi_h(x_1)\}$$

is the linear approximation of Γ_s . Let us set

$$\mathcal{V}(x_1) = v(x_1, \Psi(x_1)),$$

$$\mathcal{V}_h(x_1) = v_h(x_1, \Psi(x_1)),$$

$$\mathcal{V}_{hh}(x_1) = v_h(x_1, \Psi_h(x_1)), \quad x_1 \in \langle a,b \rangle .$$

By virtue of (3.20)

$$(3.21) \qquad \mathcal{V}_h \longrightarrow \mathcal{V} \quad \text{in } (L^2((a,b)))^2,\ h \longrightarrow 0+ .$$

Let us prove also

$$(3.22) \qquad \mathcal{V}_{hh} \longrightarrow \mathcal{V} \text{ in } (L^2((a,b)))^2,\ h \longrightarrow 0+ .$$

We may write

$$(3.23) \quad \| \mathcal{V}_{hh} - \mathcal{V} \|_{o,(a,b)} \leq \| \mathcal{V} - \mathcal{V}_h \|_{o,(a,b)} + $$

$$+ \| \mathcal{V}_h - \mathcal{V}_{hh} \|_{o,(a,b)} .$$

From the definition of $\mathcal{V}_{hh}$ it follows that these ones are piecewise linear Lagrange interpolates of $\mathcal{V}_h$ on $\langle a,b \rangle$. Corresponding division of $\langle a,b \rangle$ will be denoted by $a = t_1^m < t_2^m < \ldots < t_{m+1}^m = b$. Using the approximative property of $\mathcal{V}_{hh}$ we have

$$\| \mathcal{V}_h - \mathcal{V}_{hh} \|_{o,(a,b)} \leq ch^{1/2} \| \mathcal{V}_h \|_{1/2,(a,b)} \leq$$

$$\leq ch^{1/2} \| v_h \|_{1,\Omega} \leq ch^{1/2}$$

where $c > 0$ is independent of h for h sufficiently small and (3.19) has been used. From this, (3.21) and (3.23),(3,22) follows. Now, let us prove that

$$\int_a^b \mathcal{V}(x_1) . n(x_1, \Psi(x_1)) \varphi(x_1) \, dx_1 \leq 0$$

for any $\varphi \in C^1(\langle a,b \rangle)$, $\varphi \geq 0$ on $\langle a,b \rangle$. Using (3.22) we have

$$(3.24)\qquad \int_a^b \mathcal{V}_{hh}.n\varphi\,dx_1 \longrightarrow \int_a^b \mathcal{V}.n\varphi\,dx_1,\ h\to 0+ .$$

For the numerical computation of $\int_a^b \mathcal{V}_{hh}.n\varphi\,dx_1$ we use the trapezoid formula:

$$\int_a^b \mathcal{V}_{hh}.n\varphi\,dx_1 \approx h((\mathcal{V}_{hh}.n)(t_1^m)\varphi(t_1^m) + 2(\mathcal{V}_{hh}.n)(t_2^m)\varphi(t_2^m) + \\ + \dots + (\mathcal{V}_{hh}.n)(t_{m+1}^m)\varphi(t_{m+1}^m)) \equiv [\mathcal{V}_{hh}.n,\varphi] .$$

Since $(\mathcal{V}_{hh}.n)(t_j^m)\varphi(t_j^m) = (v_h.n)(A_j)\varphi(t_j^m) \leq 0 \quad \forall j = 1,\dots$ $\dots,m+1$ we have

$$[\mathcal{V}_{hh}.n,\varphi] \leq 0 \qquad \forall\ h>0.$$

The proof will be finished, if

$$(3.25)\qquad [\mathcal{V}_{hh}.n,\varphi] \longrightarrow \int_a^b \mathcal{V}.n\varphi\,dx_1,\ h\to 0+ .$$

We may write

$$(3.26)\qquad \left|\int_a^b \mathcal{V}.n\varphi\,dx_1 - [\mathcal{V}_{hh}.n,\varphi]\right| \leq \left|\int_a^b \mathcal{V}.n\varphi\,dx_1 - \right. \\ \left. - \int_a^b \mathcal{V}_{hh}\,n\varphi\,dx_1\right| + \left|\int_a^b \mathcal{V}_{hh}.n\varphi\,dx_1 - [\mathcal{V}_{hh}.n,\varphi]\right| .$$

By virtue of the inverse inequality between $H^{1/2}(\langle a,b\rangle)$ and $H^1(\langle a,b\rangle)$:

$$\left|\int_a^b \mathcal{V}_{hh}.n\varphi\,dx_1 - [\mathcal{V}_{hh}.n,\varphi]\right| \leq ch\,|\mathcal{V}_{hh}.n\varphi|_{1,(a,b)} \leq \\ \leq c(n,\varphi)h\,\|\mathcal{V}_{hh}\|_{1,(a,b)} \leq c(n,\varphi)h^{1/2}\|\mathcal{V}_{hh}\|_{1/2,(a,b)} \leq \\ \leq c(n,\varphi)h^{1/2}\|v_h\|_{1,\Omega_h} \leq c(n,\varphi)h^{1/2} ,$$

where Ω_h is the polygonal domain bounded with Γ_u, Γ_τ, Γ_o and Γ_{Sh}. From this, (3.19),(3.24) and (3.26) we obtain (3.25). Hence

$$(\mathcal{V}.n)(x_1) = (v.n)(x_1,\Psi(x_1)) \leq 0,\ x_1 \in \langle a,b\rangle .$$

Theorem 3.4. Let the assumptions of Lemma 3.5 and (3.4) be satisfied. Then

$$|u - u_h| \longrightarrow 0, \ h \longrightarrow 0+$$

Moreover if the solution u of ($\mathcal{P}$) is unique, then

$$\| u - u_h \|_1 \longrightarrow 0, \ h \longrightarrow 0+ \ .$$

Proof. The assertion of the Theorem is an immediate consequence of Theorem 3.2, (1.1) and Lemma 3.6.

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