Jana Ryšlinková

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ON m-ALGEBRAIC CLOSURES OF n-COMPACT ELEMENTS x)

Jana RYŠLINKOVÁ, Praha

Abstract: Let m,n be arbitrary cardinals and let u be an m-algebraic closure operator on a complete lattice. This paper answers the following question: does u preserve the n-compacticity?

Key words: n-compact element, m-directed set, m-algebraic closure operator, m-algebraic closure system.

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Introduction. The authors of the paper [3] prove that every algebraic closure operator on a complete lattice preserves the compacticity. A natural question arises: Does it preserve also the m-compacticity for any infinite cardinal m ? This paper will answer this question.

Part 1 contains only definitions and some lemmas used in part 2. A closure operator on a complete lattice is called m-algebraic, if it preserves the joins of m-directed subsets. Theorem 2.1 shows that if $m \leq n$, then

- if m is regular, then the m-algebraic closure of any n-compact element is n-compact,

- if m is irregular, then the m-algebraic closure of any n-compact element is $\max\{m^+, n\}$-compact.

Example 2.3 shows that the estimate of the compacticity

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x) This paper has originated at the seminar Algebraic Foundations of Quantum Theories. directed by Prof. Jiří Fábera.

for irregular m cannot be improved. Further, in an m-algebraic lattice $\mathcal{L}$ (where m is regular), an element is m-compact in $\mathcal{L}$ iff its m-algebraic closure is m-compact in the closure of $\mathcal{L}$.

In the whole paper, $\mathcal{L} = (L;\leq)$ will denote a given complete lattice. If $A \subseteq L$, then $\mathcal{A}$ will denote the poset $(A;\leq)$. Further, m and n denote infinite cardinals.

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1. Preliminaries.

1.1. Definition: Let m be an infinite cardinal and let $\mathcal{L} = (L;\leq)$ be a complete lattice. We shall say that $c \in L$ is m-compact in $\mathcal{L}$, if for every $X \subseteq L$ such that $c \leq \sup_{\mathcal{L}} X$, there exists $Y \subseteq X$ the cardinality of which is strictly smaller than m and such that $c \leq \sup_{\mathcal{L}} Y$.

1.2. Definition: A subset X of L is called m-directed in $\mathcal{L}$, if every subset Y of X such that $|Y| < m$ has an upper bound in X (where $|Y|$ means the cardinality of Y); more exactly:

$$(\forall Y \subseteq X)(|Y| < m \Longrightarrow (\exists x \in X)(\forall y \in Y)\ y \leq x).$$

1.3. Definition: We shall say that a mapping $u: L \longrightarrow L$ is an m-algebraic closure operator in $\mathcal{L}$ if

1) u is a closure operator in $\mathcal{L}$

2) for every m-directed subset X of L there is

$$u(\sup_{\mathcal{L}} X) = \sup_{\mathcal{L}} u(X).$$

1.4. Definition: We shall say that $A \subseteq L$ is an m-algebraic closure system in $\mathcal{L}$ if

1) A is a closure system in $\mathcal{L}$

2) for every m-directed subset X of A it holds $\sup_{\mathcal{L}} X = \sup_{(A;\leq)} X$.

1.5. Remark: a) If we set $m = \aleph_0$ in the preceding definitions 1.1 - 1.4, we obtain the usual notions of a compact element, a directed set, an algebraic closure operator or an algebraic closure system, respectively.

1.6. Lemma: (A generalization of Ward's Lemma) Let $u: L \to L$ be a closure operator on $\mathcal{L}$. Then for every $X \subseteq L$ it holds:

$$u(\sup_{\mathcal{L}} X) = \sup_{(u(L);\leq)} u(X).$$

Proof: Denote by $\mathcal{A}$ the poset $(u(L);\leq)$. Then we have $\sup_{\mathcal{A}} uX \leq u(\sup_{\mathcal{L}} X)$, since for every $x \in X$, it is $u(x) \leq u(\sup_{\mathcal{L}} X)$ and $u(\sup_{\mathcal{L}} X) \in u(L)$ is an upper bound of $u(X)$ in $\mathcal{A}$. Let for some $y \in L$, $u(y)$ be an upper bound of $u(X)$. Then $u(y)$ is also an upper bound of X, since for every $x \in X$ there is $x \leq u(x) \leq u(y)$. Therefore, $u(y) \geq \sup_{\mathcal{L}} X$, which implies $u(y) \geq u(\sup_{\mathcal{L}} X)$, i.e., $u(\sup_{\mathcal{L}} X)$ is the least upper bound of $u(X)$ in $\mathcal{A}$.

It is known that $u: L \to L$ is a closure operator iff $u(L)$ is a closure system in $\mathcal{L}$.
The following lemma shows that there is the same correspondence between m-algebraic closure operators and m-algebraic closure systems on $\mathcal{L}$:

1.7. Lemma: Let $u: L \to L$ be a closure operator. Then, for every infinite cardinal m, the following conditions are equivalent:

(1) u is an m-algebraic closure operator on $\mathcal{L}$;

(2) u(L) is an m-algebraic closure system in $\mathcal{L}$.

<u>Proof</u>: Denote by $\mathcal{A}$ the complete lattice $(u(L); \leq)$. Suppose that (1) holds and take any m-directed subset X of u(L). Since every element of X is closed, we have

$$\sup_{\mathcal{A}} X = \sup_{\mathcal{A}} u(X).$$

Further, by Lemma 1.6

$$\sup_{\mathcal{A}} u(X) = u(\sup_{\mathcal{L}} X)$$

and by assumption (1) we obtain

$$u(\sup_{\mathcal{L}} X) = \sup_{\mathcal{L}} u(X) = \sup_{\mathcal{L}} X.$$

These equalities prove that u(L) is an m-algebraic closure system in $\mathcal{L}$. Now, suppose that (2) holds and take any m-directed subset X of L. It is easy to prove that $u(X) \subseteq u(L)$ is m-directed, too. Then we have, by the assumption,

$$\sup_{\mathcal{L}} u(X) = \sup_{\mathcal{A}} u(X)$$

and by Lemma 1.6

$$\sup_{\mathcal{A}} u(X) = u(\sup_{\mathcal{L}} X).$$

These equalities show that u is m-algebraic.

1.8. <u>Lemma</u>: Let m be a regular infinite cardinal and let X be a subset of L. Then

$$X^* \underset{Df}{=} \{\sup_{\mathcal{L}} Y;\ Y \subseteq X \text{ et } |Y| < m\}$$

is m-directed and $\sup_{\mathcal{L}} X^* = \sup_{\mathcal{L}} X$.

<u>Proof</u>: Take any subset T of X^* with the cardinality strictly smaller than m. For every $t \in T$ take exactly one $Y \subseteq X$ such that $|Y| < m$ and $\sup_{\mathcal{L}} Y \in T$; denote by Z the union of all such Y.

Since cardinal m is regular and $|Z| < m$, we have

$$\sup_{\mathcal{L}} Z \in X^* .$$

Clearly, $\sup_{\mathcal{L}} Z$ is an upper bound of T and so, X^* is m-directed.

Now, denote by $\mathfrak{X}$ the set

$$\mathfrak{X} = \{ Y \subseteq X;\ \sup_{\mathcal{L}} Y \in X^* ,\ |Y| < m \} .$$

Then

$$\sup_{\mathcal{L}} X^* = \sup_{\mathcal{L}} \{ \sup_{\mathcal{L}} Y;\ Y \in \mathfrak{X} \} = \sup_{\mathcal{L}} \bigcup \mathfrak{X} = \sup_{\mathcal{L}} X$$

which proves the second assertion of the lemma.

1.9. Lemma: Let m be an irregular (infinite) cardinal and X a subset of L. Then X is m-directed if and only if X is m^+-directed. (m^+ is the cardinal successor of m.)

Proof: If X is m^+-directed it is, of course, m-directed, too. Let us suppose that X is m-directed and Y is a subset of X with the cardinality strictly smaller than m^+, i.e. $|Y| \leq m$. We have to prove that Y has an upper bound in X.

If $|Y| < m$, there is nothing to prove.

If $|Y| = m$, then there exists a family $\{Y_i; i \in I\}$ of subsets of Y such that $|I| < m$, $Y = \bigcup_{i \in I} Y_i$ and for every $i \in I$, $|Y_i| = m_i < m$. Since X is m-directed, we can choose an upper bound x_i of Y_i for every $i \in I$.

Denote by Z the set of all such x_i. Then

$|Z| \leq |I| < m$, thus Z has an upper bound $x \in X$. Clearly, x is an upper bound of Y.

2. m-algebraic closures of n-compact elements.

2.1. Theorem: Let m, n be infinite cardinals such that $m \leq n$. If $u: L \longrightarrow L$ is an m-algebraic closure operator on a complete lattice $\mathcal{L} = (L; \leq)$ and if $c \in L$ is an n-compact element of $\mathcal{L}$, then the following assertions hold:

(i) if m is regular, then u(c) is n-compact in $(u(L);\leq)$;

(ii) if m is irregular, then u(c) is $\max\{m^+, n\}$-compact in $(u(L);\leq)$.

<u>Proof</u>: Let us denote by $\mathcal{A}$ the complete lattice $(u(L;\leq)$ and by $\overrightarrow{m}$ the smallest regular cardinal α such that $m\leq \leq \alpha$. (I.e. if m is regular, then $\overrightarrow{m} = m$, and, for irregular m, $\overrightarrow{m} = m^+$.)

Let X be a subset of u(L) such that $u(c)\leq \sup_{\mathcal{A}} X$. Put

$$X^* = \{\sup_{\mathcal{L}} Y;\ |Y| < \overrightarrow{m} \text{ et } Y\subseteq X\}.$$

Then by Lemmas 1.6 and 1.8 we have $\sup_{\mathcal{A}} uX^* = \sup_{\mathcal{A}} X$. Further, u(L) is an m-algebraic closure system and the set uX^* is m-directed by Lemma 1.9; hence we obtain

$$(1) \qquad \sup_{\mathcal{A}} uX^* = \sup_{\mathcal{L}} X^* .$$

The mapping $u:L \to L$ is a closure operator, thus

$$(2) \qquad c\leq u(c)\leq \sup_{\mathcal{A}} X$$

and so, by (1) and (2) we obtain

$$c\leq \sup_{\mathcal{L}} uX^* .$$

The element x is, by the assumption, n-compact (where $m\leq n$), i.e. there exists a subset Z of uX^* the cardinality of which is strictly smaller than n and such that

$$c\leq \sup_{\mathcal{L}} Z.$$

Therefore

$$u(c)\leq u(\sup_{\mathcal{L}} Z) = \sup_{\mathcal{A}} Z.$$

For every $z\in Z$ choose exactly one $Y\subseteq X$ such that $\sup_{\mathcal{A}} Y = z$

and $|Y| < \vec{m}$. The set of all such Y will be denoted by $\mathfrak{Z}$. Then, for $X' = \bigcup \mathfrak{Z} \subseteq X$ it holds:

$$u(c) \leq \sup_{\mathcal{A}} Z = \sup_{\mathcal{A}} \{ \sup_{\mathcal{A}} Y; Y \in \mathfrak{Z} \} = \sup_{\mathcal{A}} \bigcup \mathfrak{Z} = \sup_{\mathcal{A}} X'.$$

Moreover, $|\mathfrak{Z}| \leq |Z| < n$ and $|X'| \leq \sum_{Y \in \mathfrak{Z}} |Y|$.

(i) If m is regular, we have $\vec{m} = m$ and

- for $m < n$, $\sum_{Y \in \mathfrak{Z}} |Y| \leq m \cdot |\mathfrak{Z}| < n$,
- for $m = n$, $\sum_{Y \in \mathfrak{Z}} |Y| < m = n$ by the regularity of m.

This proves assertion (i) of the theorem.

(ii) If m is irregular, we have $\vec{m} = m^+ > m$ and

- for $m < n$, there is $m^+ \leq n$, and

  $$\sum_{Y \in \mathfrak{Z}} |Y| \leq m . |\mathfrak{Z}| < n = \max\{m^+, n\},$$
- for $m = n$, there is $m^+ > n$ and

  $$\sum_{Y \in \mathfrak{Z}} |Y| \leq m . |\mathfrak{Z}| < m^+ = \max\{m^+, n\}.$$

This proves assertion (ii) of the theorem.

2.2. Example: For regular cardinal m, the assumption $m \leq n$ in Theorem 2.1 (i) cannot be omitted as shown by the following example: Put $m = \aleph_1$ (then m is regular), $n = \aleph_0$, and let N denote the set of all non-negative integers, $L = N \cup \{\omega_0, \omega_0 + 1\} \cup \{c\}$, where c is an arbitrary element which does not belong to $N \cup \{\omega_0, \omega_0 + 1\}$ and define an ordering on L as follows: the set $N \cup \{\omega_0, \omega_0 + 1\}$ is ordered by the usual ordering and for any $x, y \in L$, $x \neq c \neq y$, there is $x \leq c$ iff $x = 0$, and $c \leq y$ iff $y = \omega_0 + 1$. (See Fig.1.)

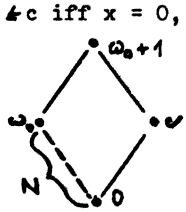

Fig.1.

Further, put $A = L \setminus \{\omega_0\}$. Then A is an $\aleph_1$-algebraic closure system in $\mathcal{L} = (L; \leq)$, since $\aleph_1$-directed subsets of A are exactly those sets $X \subseteq A$, which satisfy the condition

$$\sup_{\mathcal{L}} X \in X,$$

and so, for any $\aleph_1$-directed set $X \subseteq A$ there is

$$\sup_{\mathcal{L}} X = \sup_{\mathcal{A}} X.$$

(Here, $\mathcal{A} = (A; \leq)$.)

(Note that A is not an $\aleph_0$-algebraic closure system: e.g. N is directed, but $\sup_{\mathcal{L}} N = \omega_0 \neq \omega_0 + 1 = \sup_{\mathcal{A}} N$.)

Denote by u the $\aleph_1$-algebraic closure operator corresponding to A. Then $z(c) = c$ and c is compact in $\mathcal{L}$ but it is not compact in $\mathcal{A}$:

$$c \leq \sup_{\mathcal{A}} N = \omega_0 + 1,$$

but for any finite subset X of N, $\sup_{\mathcal{A}} X \in N$, i.e. $\sup_{\mathcal{A}} X$ is either incomparable with c or strictly smaller than c.

The following example shows that the estimate in assertion (ii) of Theorem 2.1 cannot be improved:

2.3. Example: Let m be any infinite irregular cardinal. Then there exists a complete lattice $\mathcal{L} = (L; \leq)$, an m-algebraic closure operator $u: L \longrightarrow L$ and an element $b \in L$, m-compact in $\mathcal{L}$, which is not m-compact in $(u(L); \leq)$. Of course, it is m^+-compact in $(u(L); \leq)$ by Theorem 2.1.

Let us construct the set L: since the cardinal m is infinite and irregular, there exists a limit ordinal α such that $m = \aleph_\alpha$. Denote by I the set of all ordinals $\beta < \alpha$ such that $\aleph_\beta$ is regular. This set is not empty, since $\omega_0 \in I$. Take a family of posets $\{(B_\beta; \leq_\beta)\}_{\beta \in I}$ such that the ordinal type of $(B_\beta; \leq_\beta)$ i ω_β and if $\beta, \gamma \in I$, $\beta \neq \gamma$, then $B_\beta \cap B = \emptyset$. On the other hand, take a set M, with $|M| = m$ and such that exp $M \cap B_\beta = \emptyset$ for every $\beta \in I$. Further, take two different elements b, 1 such that

$b \notin \bigcup_{\beta \in I} B_\beta \cup \exp M$, $1 \notin \bigcup_{\beta \in I} B_\beta \cup \exp M$ and put $L = \bigcup_{\beta \in I} B \cup \cup \{b\} \cup \exp M \cup \{1\}$. The ordering on L will be defined as follows: let $\mathcal{B} = (\sum_{\beta \in I} B_\beta) \oplus \{b\}$, $\mathcal{C} = (\exp M - \{\emptyset\}; \subseteq)$.

Then

$$\mathcal{L} = \{\emptyset\} \oplus (\mathcal{B} + \mathcal{C}) \oplus \{1\},$$

where $\oplus$ denotes the ordinal sum and $\sum$ or + the cardinal one. (See Fig.2.)

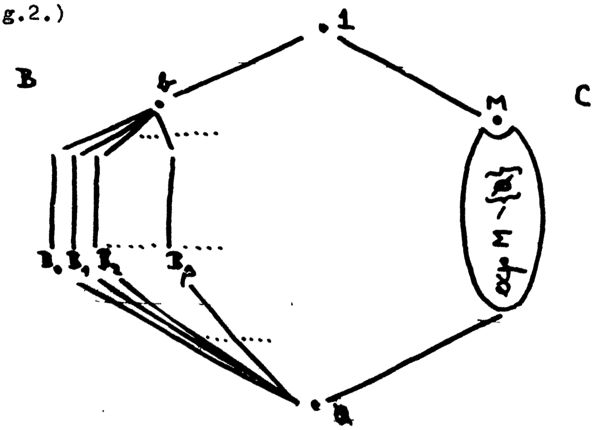

Fig.2.

It is easy to prove that $\mathcal{L}$ is a complete lattice. Put

$$A = L - \{b, M\}.$$

Then A is an m-algebraic closure system in $\mathcal{L}$: Take any m-directed subset $T \subseteq A$. We shall prove that $\sup_{\mathcal{L}} T = \sup_{\mathcal{A}} T$ (i.e. $\sup_{\mathcal{L}} T \in A$).

Suppose the contrary, $\sup_{\mathcal{L}} T \notin A$. Then either $\sup_{\mathcal{L}} T = b$ or $\sup_{\mathcal{L}} T = M$.

1) Suppose $\sup_{\mathcal{L}} T = b$. Then necessarily $T \subseteq B$. If there exists $\beta \in I$ such that $T \subseteq B_\beta$, then $|T| \leq |B_\beta| = \aleph_\beta < \aleph_\alpha = m$. Since T is m-directed, then $\sup_{\mathcal{L}} T = b \in T$ - a contradiction, because we have supposed $T \subseteq A = L - \{b, M\}$. If there

exist at least two different elements β, γ of I such that $t_1 \in T \cap B_\beta$, $t_2 \in T \cap B_\gamma$; then $|\{t_1, t_2\}| = 2 < m$, but only b and 1 are upper bounds of this subset - a contradiction again.

2) Suppose $\sup_{\mathcal{L}} T = M$. Then T must be a subset of exp M, i.e. $\sup_{\mathcal{L}} T = \bigcup T = M$. Since the cardinality of M is m (recall m is irregular), we can write $M = \bigcup_{k \in K} M_k$, where $|K| < m$ and $|M_k| < m$ for every $k \in K$. We have

$$\bigcup T = M = \bigcup_{k \in K} M_k.$$

Thus, for every $k \in K$, there exists a subset $T_k \subseteq T$ such that $|T_k| < m$ and $M_k \subseteq \bigcup T_k$. Since T is m-directed, there exists $X_k \in T$ such that $\bigcup T_k \subseteq X_k$. Put $\mathfrak{X} = \{X_k; k \in K\}$. Then $|\mathfrak{X}| < m$, $\mathfrak{X} \subseteq T$ and therefore, $\mathfrak{X}$ must have an upper bound in T, say X; a contradiction to $M = X \in T$. Hence we have proved that A is an m-algebraic closure system.

Similarly, one can prove that b is m-compact in $\mathcal{L}$. (But it is not n-compact in $\mathcal{L}$ for any $n < m$: Let $\beta \in I$ be an ordinal such that $n \leq \aleph_\beta < \aleph_\alpha$. Then $\sup_{\mathcal{L}} B_\beta = b$, but no cofinal subset of B has the cardinality smaller than n.)

Denoting by u the m-algebraic closure operator corresponding to A, we get u(b) = 1 and 1 is not m-compact in $\mathcal{A}$: for example, consider the set $\mathfrak{M} = \{\{x\}; x \in M\}$. $\mathfrak{M} \subseteq A$ and $\sup_{\mathcal{A}} \mathfrak{M} = 1$. Further, the join of every proper subset $\mathfrak{X} \subsetneq \mathfrak{M}$ is a proper subset of M and thus it is different from 1.

This completes the proof.

Theorem 2.1 characterizes m-algebraic closures of n-compact elements for some cardinals m, n. In some lattices we can characterize all n-compact elements of a closure system, as shown by the following theorems.

2.4. Theorem: Let $u:L \longrightarrow L$ be a closure operator on $\mathcal{L}$ and $\emptyset \neq C \subseteq L$ be a join-m-subsemilattice of $\mathcal{L}$ which generates L by joins. If $b \in u(L)$ is an n-compact element in $(u(L);\leq)$ for some $n \leq m$, then there exists $c \in C$ such that $b = u(c)$.

Proof: Denote by $\mathcal{A}$ the lattice $(u(L);\leq)$. There exists a set $B \subseteq C$ such that $b = \sup_{\mathcal{L}} B$, i.e.

$$b \leq u(b) = u(\sup_{\mathcal{L}} B) = \sup_{\mathcal{A}} u(B)$$

(for the latest equality use Lemma 1.6). Suppose that b is n-compact in $\mathcal{A}$; we get $B' \subseteq B$ such that $|B'| < n$ and

$$(4) \qquad b \leq \sup_{\mathcal{A}} uB'.$$

Put $c = \sup_{\mathcal{L}} B'$. Then $c \in C$ because $n \leq m$ and C is, by the assumption, a join-m-subsemilattice, thus (4) expresses the same as

$$(5) \qquad b \leq u(c).$$

On the other hand, we have

$$c = \sup_{\mathcal{L}} B' \leq \sup_{\mathcal{L}} B = b$$

and therefore,

$$(6) \qquad u(c) \leq u(b) = b.$$

Inequalities (5) and (6) prove the theorem.

2.5. Lemma: If m is a regular cardinal, then the set C of all m-compact elements of $\mathcal{L}$ is a join-m-subsemilattice.

Proof: Take any $X \subseteq C$ with the cardinality strictly smaller than m and denote by a its supremum in $\mathcal{L}$. We shall prove that a is m-compact, i.e. $a \in C$.

Let $a \leq \sup_{\mathcal{L}} Y$ where $Y \subseteq L$. Then for every $x \in X$, $x \leq \sup_{\mathcal{L}} Y$ and since X is a subset of C, then for every $x \in X$, there exists $Y_x \subseteq Y$ such that $|Y_x| < m$ and $x \not\leq \sup_{\mathcal{L}} Y_x$. Put

$$Z = \bigcup \{ Y_x ; x \in X \}.$$

By the regularity of m, we have $|Z| < m$ and, obviously, $a \not\leq \sup_{\mathcal{L}} Z$.

2.6. <u>Corollary</u>: Let m be a regular cardinal, let $\mathcal{L}$ be an m-algebraic lattice and let $u: L \rightarrow L$ be an m-algebraic closure operator. Then $a \in u(L)$ is m-compact in $(u(L); \leq)$ iff $a = u(c)$ for some element c m-compact in $\mathcal{L}$.

<u>Proof</u> follows immediately from Theorems 2.1, 2.4 and Lemma 2.5.

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Katedra matematiky FEL ČVUT
Suchbátarova 2
16627 Praha
Československo

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