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AN EXISTENCE THEOREM VIA AN INTUITIVE IDEA AND FIXED POINT THEOREMS

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Abstract: An abstract existence theorem is formulated via a simple intuitive idea, and then various fixed point theorems are deduced to illustrate the concept of abstraction in mathematics.

Key words: Existence theorem, metric space, Hausdorff metric, fixed point theorem, common fixed point theorem.

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1. Introduction and main results. A system of interest in sciences usually can be considered as a mapping S from the space X of input into the space Y of output. One of the main concerns in the study of a system is to know whether a particular desired output can be obtained in the system. The study usually starts with doing some experiments or observations about the system. A certain mean is used to measure the "distances" from the output obtained in the experiments or observations to the particular output one desires to have. Doing enough experiments or observations, one may conclude that the system should behave or can be controlled to behave in a certain manner. Suppose that the system S concerned behaves in the following manner:

For each input x and the corresponding output $S(x)$, there always exists in a certain way (cf. the condition (1) below) a new input Δx such that the corresponding output $S(\Delta x)$ is somehow "closer" to a desired particular output than the given output $S(x)$ is (cf. the condition (2) below). Then intuitively one would conclude that the desired particular output should be obtainable by a certain approximation process. This intuitive idea leads us to the formulation of the following fundamental result.

Theorem 1. Let (X, ρ) be a (non-empty) metric space, Y a non-empty set, S a mapping from X into Y, and let e be a function from $S(X)$ into the non-negative real numbers $[0,+\infty)$. Suppose that for each $x \in X$ there exists an element $\Delta x \in X$ such that the following inequalities hold:

(1) $\rho(x, \Delta x) \leq \alpha(S(x))e(S(x))$,

(2) $e(S(\Delta x)) \leq \beta(e(S(x))e(S(x))$,

where α is a bounded real-valued function defined on $S(X)$ and β is a monotone increasing function from $[0,+\infty)$ into $[0,1)$. Then starting from any $x_1 \in X$, a Cauchy sequence $\{x_n\}$ in X can be constructed such that

(3) $\lim e(S(x_n)) = 0$.

Suppose furthermore that the space (X, ρ) is complete and the mapping $e(S(\cdot))$ is continuous on X. Then for each $x_1 \in X$ there exists $u \in X$ such that

(4) $\rho(x_1, u) \leq Me(S(x_1))/[1-\beta(e(S(x_1)))]$,

and

(5) $e(S(u)) = 0$,

where M is a bound for the function α.

Here, the function e can be thought as a certain evaluation for the output of the system S so that "e(S(u)) = 0" means that S(u) is a desired particular output.

As an application of theorem 1, we give the following theorem 2, of which the proof will show a way how to apply theorem 1 by suitably defining an evaluation function. Another direct application of theorem 1 will be given at the end of the last section, where some fixed point theorems as simple consequences of theorem 2 are also given.

Theorem 2. Let (X,ρ) be a (non-empty) complete metric space, (E,d) a (non-empty) metric space, and let $P(E)$ denote the collection of all non-empty bounded closed subsets of E, D the Hausdorff metric generated by the metric d and d^* the distance function defined by

$$d^*(b,A) = \inf\ (d(b,a):a\in A),$$

where $b\in E$ and $A\in P(E)$. Let f be a continuous mapping from (X,ρ) into $(P(E),D)$ and g a continuous mapping from (X,ρ) into (E,d). Suppose that for each $x\in X$ there exists $\Delta x\in X$ such that the following inequalities hold:

$$(6)\quad \rho(x,\Delta x) \le \alpha(f(x),g(x))d^*(g(x),f(x)),$$

$$(7)\quad d^*(g(\Delta x),f(\Delta x)) \le \beta(d^*(g(x),f(x))d^*(g(x),f(x)),$$

where α is a bounded function from $g(X)\times f(X)$ into $[0,+\infty)$ and β is a monotone increasing function from $[0,+\infty)$ into $[0,1)$. Then for each $x_1\in X$ there exists an element $u\in X$ such that

$$(8)\quad \rho(x_1,u)\le Md^*(g(x_1),f(x_1))/[1-\beta(d^*(g(x_1),f(x_1))]$$

and

$$(9)\quad g(u)\in f(u),$$

where M is a bound for the function α.

We end this section by remarking that theorem 2 is an analogue of some interesting results by Fan in [4], where conditions involving certain convex structures were considered.

2. Proof of theorem 1. Let $x_1 \in X$ be fixed. Then, inductively, one constructs a sequence $\{x_n\} \subseteq X$ such that

(10) $\rho(x_n, x_{n+1}) \leq \alpha(S(x_n))e(S(x_n))$

and

(11) $e(S(x_{n+1})) \leq \beta(e(S(x_n)))e(S(x_n))$

for $n = 1,2,3,\ldots$.

For convenience, write $e_n = e(S(x_n))$ and $\beta_n = \beta(e_n)$ for $n = 1,2,3,\ldots$. Then it follows from (11) that e_n is a monotone decreasing sequence of non-negative numbers and hence e_n converges to a non-negative number, say e_o. Then $e_o \leq e_n$ for $n = 1,2,3,\ldots$. Furthermore, since β is monotone increasing, it follows from (10) and (11) that one has

(12) $\rho(x_n, x_{n+1}) \leq M\beta_1^{n-1}e_1$

and

(13) $e_o \leq e_{n+1} \leq \beta_n e_n \leq \beta_1 e_n$.

It is clear that (12) implies that $\{x_n\}$ is a Cauchy sequence and (13) implies that $e_o \leq \beta_1 e_o$ so that $e_o = 0$ and hence (3) holds.

In case that (X, ρ) is complete, the Cauchy sequence $\{x_n\}$ converges to an element, say u, in X. Then it follows from (12) that

$$\rho(x_1,u) \leq \rho(x_1,x_2)+\rho(x_2,x_3)+\ldots+\rho(x_n,x_{n+1})+\rho(x_{n+1},u)$$

$$\leq (1+\beta_1+\beta_1^2+\ldots+\beta_1^{n-1})Me_1 + \rho(x_{n+1},u)$$

$$\leq Me_1/(1-\beta_1)+\rho(x_{n+1},u)$$

for all $n = 1,2,3,\ldots$. Hence (4) holds since $\lim \rho(x_{n+1}, u) = 0$. The equality (5) follows from the equality (3) and the assumption that $e(S(\cdot))$ is continuous.

3. Proof of theorem 2. To prove theorem 2, we first prove the following elementary lemma concerning the metric d, the Hausdorff metric D and the distance function d^* as given in the theorem.

Lemma. Let $u,v \in E$ and $A,B \in P(E)$. Then

(i) $d^*(u,B) \leq d(u,v) + d^*(v,B)$,

(ii) $d^*(u,A) \leq d^*(u,B) + D(A,B)$,

(iii) $d^*(u,A) - d^*(v,B) \leq d(u,v) + D(A,B)$.

Proof of the lemma. (I). To prove (i), choose $b_n \in B$ such that $d^*(v,B) = \lim d(v,b_n)$. Then $d^*(u,B) \leq d(u,b_n) \leq d(u,v) + d(v,b_n)$ for all n, so that (i) hold.

(II). To prove (ii), choose $b_n \in B$ such that $d^*(u,B) = \lim d(u,b_n)$. Then for a given $\varepsilon > 0$, there exist $a_n \in A$ such that

$$d(a_n,b_n) \leq D(A,B) + \varepsilon$$

for $n = 1,2,3,\ldots$. Then

$$d^*(u,A) \leq d(u,a_n) \leq d(u,b_n) + d(a_n,b_n)$$

$$\leq d(u,b_n) + D(A,B) + \varepsilon$$

for all $n = 1,2,3,\ldots$. Hence

$$d^*(u,A) \leq d^*(u,B) + D(A,B) + \varepsilon .$$

Then (ii) holds since ε is arbitrary.

(III). The inequality (iii) follows immediately from (i) and (ii).

Now, we come to the proof of theorem 2. To apply theorem 1, let $Y = E \times P(E)$ and define S and e as follows:

$$S(x) = (g(x), f(x)), e(S(x)) = d^*(g(x), f(x))$$

for all $x \in X$. Note that by (iii) in the lemma one has, for all $x, y \in X$,

$$e(S(x)) - e(S(y)) \leq d(g(x), g(y)) + D(f(x), f(y)).$$

The above inequality remains true when x and y are interchanged. Hence for all $x, y \in X$ one has

$$|e(S(x) - e(S(y))| \leq d(g(x), g(y)) + D(f(x), f(y)),$$

and from which one concludes that the function $e(S(\cdot))$ is continuous on X. Therefore, by theorem 1 for each $x_1 \in X$ there exists $u \in X$ such that (8) holds and also

$$d^*(g(u), f(u)) = e(S(u)) = 0,$$

from which one sees that (9) holds since f(u) is closed.

4. Fixed point theorems. In theorem 2, if the space X happens to be a subspace of E, taking g to be the identity mapping on X, we have the following fixed point theorem.

Theorem 3. Let (E,d) be a metric space, X a non-empty complete and closed subspace of E, f a continuous mapping from (X,d) into (P(E),D). Suppose that for each $x \in X$, there exists $\Delta x \in X$ such that

$$(14) \quad d(x, \Delta x) \leq \alpha(x, f(x)) d^*(x, f(x)),$$

$$(15) \quad d^*(\Delta x, f(\Delta x)) \leq \beta(d^*(x, f(x))) d^*(x, f(x)),$$

where α, β are as given in theorem 2. Then for each $x_1 \in X$, there exists an element $u \in X$ such that

(16) $d(x_1,u) \leq Md^*(x_1,f(x_1))/\{1-\beta(d^*(x_1,f(x_1)))\}$

and

$u \in f(u)$

or

$u = f(u)$ provided that f is single-valued.

Theorem 3 seems to be interesting since the mapping f is not required to be a contraction or an <u>inner</u> mapping (i.e. a mapping with images lying in its domain or the power set of its domain). Fixed point theorems for mappings which are not necessarily inner date back to [9], and recently many new results related to the concept of inwardness have appeared (see [1],[2],[5],[6],[7],[8]). Note that theorem 3 does not involve any of the known inwardness conditions, and is clearly a generalization of the existence portion of the well-known Banach contraction principle. For inner mappings, we list the following result, which is a variant of Sehgal's result [10].

<u>Theorem 4</u>. Let (X,d) be a non-empty complete metric space, f a continuous mapping from X into X. Suppose that for each $x \in X$, there exists a positive integer $n = n(x)$ such that

$d(x,f^n(x)) \leq \alpha(x,f(x))d(x,f(x))$

and

$d(f^n(x),f^{n+1}(x)) \leq \beta(d(x,f(x)))d(x,f(x))$,

where α and β are as given in theorem 3. Then for each $x_1 \in X$, there exists $u \in X$ such that the inequality (16) holds and $u = f(u)$.

Proof. Take $\Delta x = f^n(x)$ and apply theorem 3.

Viewing theorem 3 in a slight different angle, we have the following common fixed point theorem.

Theorem 5. Let (E,d) be a metric space, X a non-empty complete and closed subspace of E, f a continuous mapping from X into E and Δ a mapping (not necessarily continuous) from X into X. Suppose that the following inequalities hold for all $x \in X$:

(17) $d(x,\Delta x) \leq \alpha(d(x,f(x)))d(x,f(x))$,

(18) $d(f(\Delta x),\Delta x) \leq \beta(d(x,f(x))d(x,f(x))$.

Then both f and Δ have the same non-empty set of fixed points.

Proof. By theorem 3, f has fixed points in X. Let u be a fixed point of f. Then (17) implies that $d(u,\Delta u) = 0$ so that u is a fixed point of Δ. Let v be a fixed point of Δ. Then by (18) one has

$d(f(v),v) \leq \beta(d(v,f(v)))d(v,f(v))$,

which implies that $d(f(v),v) = 0$, and hence v is a fixed point of f, completing the proof.

For common fixed points of a family of commuting inner mappings, we give the following result.

Theorem 6. Let (X,d) be a non-empty complete metric space, and $\mathcal{F}$ a family of commuting mappings from X into X. Suppose that there exists a mapping $f \in \mathcal{F}$ satisfying all the conditions listed in theorem 4 and suppose furthermore that the mapping f is contractive (i.e. $d(f(x),f(y)) < d(x,y)$ for all $x,y \in X$ with $x \neq y$). Then the family $\mathcal{F}$ has a unique common fixed point.

We remark that the notion of contractive mappings related to fixed point theory has been initially studied by Edelstein in [3], where many other interesting notions have been introduced.

Proof of Theorem 6. By theorem 4, the mapping f has a fixed point and hence f has a unique fixed point since f is contractive. Let u be the unique fixed point of f and let $g \in \mathcal{F}$. Then if u were not a fixed point of g, one would have

$$\begin{aligned} d(g(u),u) &= d(g(f(u)),f(u)) \\ &= d(f(g(u)),f(u)) < d(g(u),u), \end{aligned}$$

which is impossible. Hence u is the unique common fixed point of the family $\mathcal{F}$.

Various types of results concerned common fixed points can be obtained by directly applying the fundamental result, theorem 1. As an example, we give the following results concerning a family of finitely many mappings which are not necessarily inner or single-valued. Here P(E),D, and d^* are the same as those given in theorem 2.

<u>Theorem 7</u>. Let (E,d) be a metric space, X a non-empty complete closed subspace of E, f_i a continuous mapping from (X,d) into (P(E),D) for $i = 1,2,3,\ldots,n$, and let φ be a function defined on X by either

$$\varphi(x) = \sum d^*(x,f_i(x))$$

or

$$\varphi(x) = \max d^*(x,f_i(x)),$$

where both Σ and max are taking for $i = 1,2,3,\ldots n$. Suppose that for each $x \in X$, there exists $\Delta x \in X$ such that the following hold:

(19) $d(x,\Delta x) \leq \alpha(x)\varphi(x)$,

(20) $\varphi(\Delta x) \leq \beta(\varphi(x))\varphi(x)$,

where α is a bounded real-valued function defined on X and β is a monotone increasing function from $[0,+\infty)$ into $[0,1)$. Then there exists an element $u \in X$ such that

$u \in f_i(u)$ for all $i = 1,2,3,\dots,n$.

Proof. Similar to the proof of theorem 2, one shows that $d^*(\cdot, f_i(\cdot))$ is continuous on X and hence so is the function φ. Define

$Y = X \times [P(E)]^n$,

$S(x) = (x, f_1(x), f_2(x), \dots, f_n(x))$,

$e(S(x)) = \varphi(x)$

for all $x \in X$ and apply theorem 1 to obtain an $u \in X$ such that $\varphi(u) = 0$. Then

$d^*(u, f_i(u)) = 0$ so that $u \in f_i(u)$ for $i = 1,2,3,\dots,n$.

We end this note by remarking that the recent book by Smart [11] gave a very good introduction to various fixed point theorems and to their applications in analysis.

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