Commentationes Mathematicae Universitatis Carolinae

H. L. Bentley; Horst Herrlich

Completion as reflection

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 3, 541--568

Persistent URL: http://dml.cz/dmlcz/105875

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,3 (1978)

COMPLETION AS REFLECTION

H.L. BENTLEY and H. HERRLICH

Abstract: Completeness is known to be reflective in the category of uniform T_1-spaces and in the larger category of regular nearness T_1-spaces. In this paper the even larger category SepNear of separated nearness T_1-spaces is investigated. It is shown that SepNear is epireflective in Near and that completeness is reflective in SepNear. As opposed to the uniform and regular case, the complete reflection in the separated case is not the strict completion but is the simple completion, introduced in this paper. The strict completion can still be regarded as a complete reflection, if attention is restricted to those maps which have uniformly continuous extensions to the strict completions. These maps are characterized internally. As an application, a characterization is given of those maps between Hausdorff topological spaces, which can be continuously extended to the Katětov - resp. Fomin-H-closed extensions.

Key words: Completeness; strict and simple completion; uniform spaces; separated nearness spaces; reflective subcategories; H-closed extensions; (uniformly) continuous extensions of maps.

AMS: 54E15, 54D35

Introduction and summary. As is well-known, every uniform space can be densely embedded into a complete uniform space and - if attention is restricted to separated uniform spaces - this completion can be considered as an epireflection. The concept of uniform spaces has been generalized to that of nearness spaces in order to include all symmetric topological spaces, and it has been shown that every nearness space X can be strictly embedded into a complete nearness space X^* , called its strict completion. The question, whether this completion can be considered as an epireflection, has found a positive answer for regular nearness spaces (Morita [23], Herrlich [16], see also Steiner and Steiner [29]). In this paper, the following two questions will be answered:

(A) Do the complete, separated nearness spaces form an epireflective subcategory of the category of all separated nearness spaces?

(B) Can the strict completion $X \longrightarrow X^*$ be regarded as an epireflection for separated nearness spaces?

Surprisingly, the answer to (A) is yes and the answer to (B) is no and yes. A proper understanding of these questions requires a careful analysis of separated nearness spaces first.

In § 1, we will show that the concept of separated nearness spaces has the following properties:

(1) Every regular (and hence in particular every uniform) nearness space is separated.

(2) A topological space is Hausdorff iff it is separated as a nearness space.

(3) Separatedness is productive and hereditary; hence the separated nearness spaces form an epireflective subcategory of all nearness spaces.

(4) Separatedness is preserved by the strict completion.

(5) Separatedness is preserved by the topological coreflection, i.e. the underlying topological space of a se-

parated nearness space is always Hausdorff.

In § 2, we give a positive answer to the above Problem (A) and provide a concrete description of the complete reflection $X \longrightarrow \tilde{X}$ of a separated nearness space X, which will be called the simple completion of X.

Since the simple completion behaves rather badly in so far as it destroys any of the properties "uniform", "regular", "contigual", "proximal", whereas the strict extension preserves all of them, we investigate in § 3 the question whether the strict completion can be regarded as a reflection in some modified category. The answer is yes, if we restrict our attention to the category of all separated nearness spaces and those maps $f: X \longrightarrow Y$ which have a uniformly continuous (= nearness preserving) extension $f^{*}: X^{*} \longrightarrow Y^{*}$ to the strict completions. We characterize these "extendable" maps and show, in particular, that uniformly continuous maps $f: X \longrightarrow Y$ are extendable provided any of the following conditions hold:

(1) X is complete.

(2) Y is regular.

(3) f is a projection.

As an application of the above results, in § 4 we characterize those maps between Hausdorff topological spaces which can be extended continuously to the Katětov or the Fomin H-closed extensions.

In this paper, all spaces (nearness or topological or uniform) are supposed to be T_1-spaces.

0. <u>Background</u>. A <u>nearness space</u> X is a pair consisting of a set - called the <u>underlying set of</u> X and, par abuse de language, usually also denoted by X - and a non-empty collection of non-empty covers of the set X - called <u>uniform covers of</u> X - satisfying the following axioms:

(1) Any cover of X refined by some uniform cover of X is itself uniform.

(2) Any two uniform covers $\mathcal{A}$ and $\mathcal{B}$ of X have a common uniform refinement.

(3) If $\mathcal{A}$ is a uniform cover of X, then so is $\mathrm{int}_X\mathcal{A} = \{\mathrm{int}_X A \mid A \in \mathcal{A}\}$, where $x \in \mathrm{int}_X A$ iff $\{X - \{x\}, A\}$ is a uniform cover of X.

The operator int_X defines a topology on the set X. The corresponding topological space is called the <u>underlying topological space of</u> X and all topological terms (open sets, adherence points, etc.) in a nearness space refer to this topology. In this paper, we restrict our attention to those nearness spaces X, whose underlying topological space is a T_1-space (equivalently: such that $\{X - \{x\}, X - \{y\}\}$ is a uniform cover of X for any two different points x and y of X). A map $f: X \longrightarrow Y$ between nearness spaces is called <u>uniformly continuous</u>, provided the f-preimage $f^{-1}\mathcal{A} = \{f^{-1}A \mid A \in \mathcal{A}\}$ of any uniform cover $\mathcal{A}$ of Y is a uniform cover of X. The category of nearness spaces and uniformly continuous maps will be denoted by <u>Near</u>.

A collection $\mathcal{A}$ of subsets of a nearness space X is called <u>near in</u> X provided any uniform cover contains a member which meets every member of $\mathcal{A}$ (equivalently: if

$\{X - A \mid A \in \mathcal{A}\}$ is not a uniform cover of X). Every collection with an adherence point is near in X, but generally not vice versa. A collection of subsets of X, which is not near in X, is called <u>far in</u> X. A collection $\mathcal{A}$ of subsets of X is called <u>micromeric in</u> X provided that for any uniform cover $\mathcal{B}$ of X there exists $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $A \subset B$. A micromeric filter is called a <u>Cauchy filter</u>. Every convergent filter is a Cauchy filter, but generally not vice versa. It can be seen easily that a collection $\mathcal{A}$ of subsets of X is near (resp. micromeric) in X iff the collection

$$\sec \mathcal{A} = \{B \subset X \mid B \cap A \neq \phi \text{ for all } A \in \mathcal{A}\}$$

is micromeric (resp. near) in X. If a collection $\mathcal{A}$ <u>corefines</u> $\mathcal{B}$, i.e. if $\{X - A \mid A \in \mathcal{A}\}$ refines $\{X - B \mid B \in \mathcal{B}\}$, and $\mathcal{A}$ is micromeric (resp. $\mathcal{B}$ is near), then $\mathcal{B}$ is micromeric (resp. $\mathcal{A}$ is near). A map between nearness spaces is uniformly continuous iff it preserves near collections (equivalently: iff it preserves micromeric collections).

For any nearness space X, the maximal elements of the set of all non-empty near collections in X, ordered by inclusion, are called <u>clusters in</u> X. Every cluster $\mathcal{A}$ is a <u>grill</u>, i.e. satisfies

$$A \cup B \in \mathcal{A} \quad \text{iff } (A \in \mathcal{A} \text{ or } B \in \mathcal{A}).$$

Every near grill in X is micromeric in X. A nearness space X is called complete provided every cluster has an adherence point in X. Every nearness space can be densely embedded into some complete nearness space. The embedding $e: X \longrightarrow X^*$, described below, is called the <u>strict completion</u>

_of_ X. The underlying set of X^* consists of all clusters in X. The function $e: X \longrightarrow X^*$ maps every $x \in X$ onto the cluster consisting of all those subsets of X which have x as an adherence point. For any subset B of X, the set B^* denotes the set of all $p \in X^*$ such that B meets every member of the cluster p. A cover $\mathcal{A}$ of X^* is called uniform provided there exists a uniform cover $\mathcal{B}$ of X such that $\{B^* \mid B \in \mathcal{B}\}$ refines $\mathcal{A}$.

A nearness space X is called _uniform_ (or a uniform space) provided every uniform cover of X has a uniform star-refinement. For uniform spaces, the above concepts correspond with the familiar uniform concepts. In particular, $e: X \longrightarrow X^*$ is the usual completion of a uniform space. A nearness space X is called _regular_ provided, for every uniform cover $\mathcal{A}$ of X, the collection

$$\{B \subset X \mid B <_X A \text{ for some } A \in \mathcal{A}\}$$

is also a uniform cover of X, where $B <_X A$ means that $\{A, X - B\}$ is a uniform cover of X. Every uniform space is regular. For the categories _Unif_ resp. _Reg_ of uniform resp. regular nearness spaces and uniformly continuous maps, the strict completion can be characterized as complete (epi) reflections. A nearness space X is called _topological_ provided every open cover of X is uniform. Every topological nearness space is complete. We will identify every topological (T_1-) space X with the topological nearness space X whose uniform covers are precisely the interior covers of X, i.e. those covers which can be refined by some open cover of X. (This provides, in fact, an isomorphism between the category of all topological (T_1-) spaces and continuous maps and of all topological nearness spaces and uniformly continuous maps.)

1. Separated nearness spaces. We study here the category SepNear of separated nearness spaces. Unfortunately, the definition of separated nearness spaces is technically unappealing, but if only the reader can survive reading it, we expect that he will be pleasantly rewarded by seeing some of the nice properties of the category SepNear.

1.1 Definition. A nearness space X is called separated provided that whenever a collection $\mathcal{A}$ is both near and micromeric then $\{G \subset X \mid \{G\} \cup \mathcal{A}$ is near in $X\}$ is near in X. The full subcategory of Near whose objects are the separated nearness spaces is denoted by SepNear.

The next four results appear in the papers Herrlich [17] and Bentley and Herrlich [4]; also, in the former paper, Theorem 1.6 below was incorrectly stated to be false.

1.2 Proposition. A topological space is Hausdorff iff it is separated as a nearness space. Moreover, the underlying topological space of a separated nearness space is Hausdorff.

1.3 Proposition. Every regular nearness space is separated. Hence, every uniform space is separated.

1.4 Theorem. A nearness space X is separated iff its strict completion X^* is separated.

1.5 Proposition. <u>SepNear</u> is hereditary in <u>Near</u>. I.e. any nearness subspace of a separated nearness space is separated.

1.6 <u>Theorem</u>. <u>SepNear</u> is productive in <u>Near</u>.

Proof: Let $(X_i)_{i\in I}$ be a family of separated nearness spaces and let $X = \prod_{i\in I} X_i$ be their product in <u>Near</u> with $(p_i : X \longrightarrow X_i)_{i\in I}$ the projections. Let $\mathcal{A}$ be a collection which is both near and micromeric in X. To prove that

$$\mathcal{B} = \{ B \subset X \mid \{B\} \cup \mathcal{A} \quad \text{is near in } X \}$$

is near in X, let $\mathcal{U}$ be a uniform cover of X. Then there exist a finite subset J of I and, for each $j \in J$, a uniform cover $\mathcal{U}_j$ of X_j such that x)

$$\bigwedge_{j\in J} p_j^{-1}\, \mathcal{U}_j \text{ refines } \mathcal{U} .$$

For each $i \in I$ the collection $p_i \mathcal{A}$ is both near and micromeric in X_i, hence - by separatedness of X_i - the collection

$$\mathcal{B}_i = \{ B \subset X_i \mid \{B\} \cup p_i \mathcal{A} \quad \text{is near in } X_i \}$$

is near in X_i. Consequently, for each $j \in J$, there exists a $U_j \in \mathcal{U}_j$ meeting each member of $\mathcal{B}_j$. This implies that, for each $j \in J$, $X_j - U_j \notin \mathcal{B}_j$, hence $p_j^{-1}(X_j - U_j) \notin \mathcal{B}$.

----

x) If $(\mathcal{A}_j)_{j\in J}$ is a family of collections of subsets of X then

$$\bigwedge_{j\in J} \mathcal{A}_j = \{ \bigcap_{j\in J} A_j \mid A_j \in \mathcal{A}_j \}.$$

Since $\mathscr{G}$ is a grill, no subset of

$$\bigcup_{j \in J} p_j^{-1}(X_1 - U_j) = X - \bigcap_{j \in J} p_j^{-1} U_j$$

belongs to $\mathscr{G}$, i.e. every member of $\mathscr{G}$ meets

$$\bigcap_{j \in J} p_j^{-1} U_j.$$

If U is an element of $\mathfrak{U}$ with

$$\bigcap_{j \in J} p_j^{-1} U_j \subset U$$

then U meets every member of $\mathscr{G}$. Consequently, $\mathscr{G}$ is near in X.

The above proof works not only for products but for any initial (mono-) source $(p_i : X \to X_i)_{i \in I}$ in Near. In particular, it provides a proof for 1.5 too.

1.7 Corollary.

(1) SepNear is an epireflective subcategory of Near.

(2) SepNear is complete and cocomplete.

(3) The forgetful functor SepNear $\to$ Set is (onto, mono source)-topological.

1.8 Proposition. In SepNear, the monomorphisms are precisely those uniformly continuous maps which are injective and the epimorphisms are precisely those uniformly continuous maps $f : X \to Y$ for which fX is dense in Y.

1.9 Corollary. SepNear is well-powered and cowell-powered.

2. <u>The complete reflection</u>. Herein appear our main results concerning the epireflective nature of completeness in the category <u>SepNear</u>.

2.1 Proposition. For a nearness space X, the following conditions are equivalent:

(1) X is complete and separated.

(2) Any collection, which is near and micromeric in X, has a unique adherence point in X.

Proof: (1) $\longrightarrow$ (2). If $\mathcal{A}$ is near and micromeric in X, then by separatedness,

$$\mathcal{L} = \{ B \subset X \mid \{B\} \cup \mathcal{A} \text{ is near in } X \}$$

is a cluster in X. By completeness, $\mathcal{L}$ and hence $\mathcal{A}$ has an adherence point. If x and y are adherence points of $\mathcal{A}$, then the sets $\{x\}$ and $\{y\}$ belong to $\mathcal{L}$, which implies $x = y$.

(2) $\longrightarrow$ (1). Since every cluster is near and micromeric, (2) implies completeness of X. Next, let $\mathcal{A}$ be near and micromeric in X, let x be the unique adherence point of $\mathcal{A}$, and let

$$\mathcal{L} = \{ B \subset X \mid \{B\} \cup \mathcal{A} \text{ is near in } X \} .$$

To show that $\mathcal{L}$ is near, it suffices to show that x is an adherence point of every $B \in \mathcal{L}$. If this were not true for some $B \in \mathcal{L}$, then the micromeric collection $\mathcal{A} \cup \{B\}$ would have no adherence point, and hence by (2) could not be near in X, in contradiction to the definition of $\mathcal{L}$.

2.2 Corollary. ([17]) For a separated nearness space X, the following conditions are equivalent:

(1) X is complete.

(2) Every near grill has an (unique) adherence point.

(3) Every Cauchy filter converges to a (unique) point in X.

2.3 Theorem. Completeness in SepNear is productive.

Proof: Let $(X_i)_{i\in I}$ be a family of complete, separated nearness spaces and let $X = \prod_{i\in I} X_i$ be their product in Near with $(p_i : X \longrightarrow X_i)_{i\in I}$ the projections. Let $\mathcal{A}$ be a Cauchy filter in X. Then, for each $i \in I$, $p_i\mathcal{A}$ is a Cauchy filter in X_i and converges by 2.2 to some point x_i. Consequently $\mathcal{A}$ converges to $x = (x_i)_{i\in I}$, which by 2.2 implies completeness of X.

We remark here that completeness is productive even in Near, but to prove that would require a big digression.

2.4 Proposition. Completeness is closed - hereditary in SepNear.

2.5 Corollary. Completeness is epireflective in SepNear.

2.6 Remark. Since every separated nearness space can be densely embedded into a complete separated nearness space (see 1.4), the epireflection maps are dense embeddings.

Our next objective is an internal description of the complete reflection of a given separated nearness space.

2.7 Definition. The simple completion $e : X \longrightarrow \tilde{X}$ of a nearness space X is defined as follows:

(1) $\tilde{X}$ is the set of all clusters in X.

(2) $e : X \longrightarrow \tilde{X}$ is defined by $e(x) = \{A \subset X \mid x \in \mathrm{Cl}_X A\}$.

(3) A cover $\mathcal{U}$ of $\tilde{X}$ is uniform iff it satisfies the following two conditions:

(a) $e^{-1}\mathcal{U}$ is a uniform cover of X,

(b) for each $p \in \tilde{X}$ there exists $A \in \mathcal{U}$ such that $p \in A$ and $e^{-1}A$ meets each member of the cluster p.

Any completion of X, which is isomorphic to $e:X \longrightarrow \tilde{X}$, will be called a <u>simple completion</u> of X.

2.8 Remarks. It can be seen easily that $\tilde{X}$ is a complete nearness space and $e:X \longrightarrow \tilde{X}$ is a dense embedding in <u>Near</u>, i.e. $e:X \longrightarrow \tilde{X}$ is a completion. Moreover, the following hold:

(1) eX is open in $\tilde{X}$,

(2) each point of $\tilde{X} - eX$ is isolated in $\tilde{X} - eX$,

(3) $p \in \mathrm{int}_{\tilde{X}} A$ if $p \in A$ and $e^{-1}A$ meets every member of the cluster p.

(4) $p \in \mathrm{Cl}_{\tilde{X}} A$ iff ($p \in A$ or $e^{-1}A \in p$).

(5) $\mathcal{U}$ is near in $\tilde{X}$ if $\mathcal{U}$ has an adherence point in $\tilde{X}$ or $e^{-1}\mathcal{U}$ is near in X.

(6) $\mathcal{U}$ is micromeric in $\tilde{X}$ iff $\mathcal{U}$ converges x) in $\tilde{X}$ or $\{B \subset X \mid eB \in \mathcal{U}\}$ is micromeric in X.

(7) $\mathcal{U}$ is a uniform cover of $\tilde{X}$ iff $\mathcal{U}$ is an interior cover of $\tilde{X}$ and $e^{-1}\mathcal{U}$ is a uniform cover of X.

2.9 <u>Theorem</u>. If X is separated, so is its simple completion $\tilde{X}$.

----

x) $\mathcal{U}$ converges to p in $\tilde{X}$ iff every neighborhood of p in $\tilde{X}$ contains some member of $\mathcal{U}$.

Proof: Before showing that $\tilde{X}$ is separated, we first show that the underlying topological space of $\tilde{X}$ is Hausdorff. Let p, $q \in X$ with $p \neq q$ and let

$$\mathcal{G} = \{ G \subset X \mid G \text{ meets every member of } p \}.$$

Then $\mathcal{G}$ is near and micromeric in X so, since $\mathcal{G}$ is contained in the cluster p, it cannot be contained in q. Let $G \in \mathcal{G} - q$ and let $H = X - G$. Then $\{p\} \cup eG$ and $\{q\} \cup eH$ are disjoint neighborhoods of p and q respectively.

Now, let $\mathcal{A}$ be near and micromeric in $\tilde{X}$. By 2.1, it is sufficient to show that $\mathcal{A}$ has a unique adherence point in $\tilde{X}$.

Consider first the case in which $\mathcal{A}$ converges in $\tilde{X}$. Since the underlying topological space of $\tilde{X}$ is Hausdorff, $\mathcal{A}$ can have at most one adherence point. Suppose that $\mathcal{A}$ does not have an adherence point. Then $e^{-1}\mathcal{A}$ is near in X. But $e^{-1}\mathcal{A}$ is also micromeric in X so

$$p = \{ B \subset X \mid \{B\} \cup e^{-1}\mathcal{A} \text{ is near in } X \}$$

is a cluster in X. Clearly, p is an adherence point of $\mathcal{A}$, which is a contradiction.

Next, consider the case in which $\mathcal{A}$ does not converge in $\tilde{X}$. Then

$$\mathcal{B} = \{ B \subset X \mid eB \in \mathcal{A} \}$$

is micromeric in X. But $\mathcal{B}$ is also near in X so

$$p = \{ E \subset X \mid \{E\} \cup \mathcal{B} \text{ is near in } X \}$$

is a cluster in X. In order to show that p is an adherence point of $\mathcal{A}$, let $A \in \mathcal{A}$ and suppose that $p \notin Cl_{\tilde{X}} A$. Then $e^{-1} A \notin p$ and so $\{e^{-1} A\} \cup \mathcal{B}$ is far in X. Since $\mathcal{B} \subset e^{-1}\mathcal{A}$, then $e^{-1}\mathcal{A} = \{e^{-1}A\} \cup e^{-1}\mathcal{A}$ must also be far in X. But then $\mathcal{A}$ must have some adherence point q in $\tilde{X}$. Since $e\mathcal{B} \subset \subset \mathcal{A}$, then it must be that $\mathcal{B} \subset q$. Consequently, p = q, a contradiction. This same argument can be used to show that the adherence point of $\mathcal{A}$ is unique.

2.10 Theorem. If $e:X \longrightarrow Y$ is a separated completion, then the following are equivalent:

(1) $e:X \longrightarrow Y$ is a complete reflection of X in <u>SepNear</u>.

(2) $e:X \longrightarrow Y$ is a simple completion of X.

(3) $e:X \longrightarrow Y$ is topologically a simple extension (i.e., if A is open in Y then so is any set B with $(A \cap eX) \subset B \subset A$), and the following equivalent conditions hold:

(a) Every interior cover $\mathcal{A}$ of Y, such that $e^{-1}\mathcal{A}$ is a uniform cover of X, is a uniform cover of Y.

(b) If $\mathcal{A}$ is near but has no adherence point in Y, then $e^{-1}\mathcal{A}$ is near in X.

(c) If $\mathcal{A}$ is micromeric but does not converge in Y, then $\{B \subset X \mid eB \in \mathcal{A}\}$ is micromeric in X.

Proof: $(1) \longleftrightarrow (2)$ We need only to establish the appropriate universal mapping property of the simple completion $e:X \longrightarrow \tilde{X}$. Let Z be a complete, separated nearness space and let $f:X \longrightarrow Z$ be a uniformly continuous map. Let $g:\tilde{X} \longrightarrow Z$ be defined as follows: for each $p \in \tilde{X}$, since the cluster p is near and micromeric, then so is the collection

$$fp = \{fA \mid A \in p\} .$$

Therefore, fp has a unique adherence point g(p) in Z. It is easy to show that $g \circ e = f$ and that $g:\tilde{X} \longrightarrow Z$ is uniformly continuous.

(2) $\leftrightarrow$ (3) is a simple exercise.

2.11 Remark. Even though, from a categorical point of view, the simple completion of a separated nearness space X - being its complete reflection in <u>SepNear</u> - is the "nicest" completion of X, it is not so from a more concrete point of view, since it may destroy many nice properties of X, as the following example demonstrates.

2.12 Example. Let $[0,1]$ be the closed unit interval with its usual nearnes (= uniform, = topological) structure, and let X be the nearness subspace of $[0,1]$, determined by the set $[0,1] - \{ \frac{1}{n} \mid n = 1,2,\dots \}$. Then X, as a subspace of a compact Hausdorff space, is proximal and hence uniform, regular, and contigual. But the complete reflection of X (which is nothing else but the simple topological extension of the underlying topological space TX of X, determined by the same filter traces as the strict extension $[0,1]$ of TX) is neither proximal nor uniform nor regular nor contigual. Moreover:

(1) $\tilde{X}$ is never uniform or regular, unless it coincides with the strict completion X^* of X.

(2) $\tilde{X}$ is never proximal or contigual unless $\tilde{X} - eX$ is finite.

2.13 Remark. Heldermann [13], [14] has studied the category <u>HausNear</u>, the full subcategory of <u>Near</u> whose objects are those nearness spaces whose underlying topological space is Hausdorff. That study was focused mainly on category theoretic properties of <u>HausNear</u>, e.g. a characterization of epis and monos is given.

Carlson has introduced the concept of ultrafilter complete nearness spaces (= B-complete in [6]), i.e. those nearness spaces in which every near ultrafilter has an adherence point, and has shown [7] that ultrafilter complete spaces form an epireflective subcategory of <u>HausNear</u>. Although for separated spaces, the concepts complete and ultrafilter complete coincide, (Proof: It follows from (1) $\longrightarrow$ (2) of 2.2 that complete implies ultrafilter complete. Suppose X is ultrafilter complete and let $\mathcal{A}$ be a cluster on X. Let $\mathcal{G}$ be the collection of all subsets of X which meet every member of $\mathcal{A}$. Then $\mathcal{G}$ is a near and Cauchy filter in X. Consequently, $\mathcal{A}$ consists of precisely those subsets A of X for which $\{A\} \cup \mathcal{G}$ is near in X. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{G} \subset \mathcal{U}$. Then $\mathcal{U}$ is a Cauchy filter, hence $\mathcal{U}$ is near. Therefore, $\mathcal{U}$ has an adherence point x which must also be an adherence point of $\mathcal{G}$. Consequently, $\{x\} \in \mathcal{A}$ and so x is an adherence point of $\mathcal{A}$. Therefore, X is complete.), his ultrafilter completion $X \longrightarrow X'$ is, in general, different from our simple completion $X \longrightarrow \tilde{X}$. In particular, X' need be neither complete nor separated. (Example: Let X be the nearness subspace of $[0,1]$ determined by the set $\{\frac{1}{n} \mid n = 1,2,\dots\}$. Then X' is neither complete nor separated.)

3. <u>The strict completion as reflection.</u> The strict completion preserves the properties separated, regular,

uniform, contigual, proximal and it preserves the dimension. Hence it is natural to ask whether, for separated spaces, it can be regarded as a reflection in some sensible way.

3.1 Definition. A uniformly continuous map $f:X \to Y$ between nearness spaces X and Y is called <u>extendible</u> provided there exists a uniformly continuous map $g:X^* \to Y^*$ between the strict completions of X and Y for which the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X^* \arrow[d, "g"] \\ Y \arrow[r] & Y^* \end{tikzcd}$$

commutes, where $X \to X^*$ and $Y \to Y^*$ are the strict completions of X and Y respectively.

3.2 Proposition. The strict completion is the complete reflection in the category of separated nearness spaces and extendible maps.

Proof: See Porter [24] .

3.3 Proposition. Let $f:X \to Y$ be a uniformly continuous map between separated nearness spaces X and Y. Then the following are equivalent:

(1) $f:X \to Y$ is extendible.

(2) If $\mathcal{B}$ is far in Y then there exists $\mathcal{A}$ far in X such that for each $A \in \mathcal{A}$ there exists some $B \in \mathcal{B}$ such that the following condition holds: whenever $\mathcal{G}$ is a cluster in X with $\{A\} \cup \mathcal{G}$ far in X then $\{B\} \cup f\mathcal{G}$ is far in Y.

(3) If $\mathcal{L}$ is far in Y then there exists $\mathcal{A}$ far in X such that for each $A \in \mathcal{A}$ there exists some $B \in \mathcal{L}$ such that the following condition holds: whenever $\mathcal{G}$ is both near and micromeric in X with $\{A\} \cup \mathcal{G}$ far in X then $\{B\} \cup f\mathcal{G}$ is far in Y.

Proof: (1) $\longrightarrow$ (2). Let $g: X^* \longrightarrow Y^*$ be uniformly continuous with $g \circ e_X = e_Y \circ f$ where $e_X: X \longrightarrow X^*$ and $e_Y: Y \longrightarrow Y^*$ are the strict completions of X and Y respectively. Let $\mathcal{L}$ be far in Y. Then $f^{-1} Cl_{Y*}\, e_Y \mathcal{L}$ is far in X^*. So, by construction of X^*,

$$\mathcal{A} = \{A \subset X \mid \text{for some } H \in g^{-1}\, Cl_{Y*}\, e_Y \mathcal{L},\ H \subset Cl_{X*}\, e_X A\}$$

is far in X. For each $A \in \mathcal{A}$ there exists $B \in \mathcal{L}$ with $g^{-1} Cl_{Y*}\, e_Y B \subset Cl_{X*}\, e_X A$. Let $\mathcal{G}$ be a cluster in X with $\{A\} \cup$ $\cup\, \mathcal{G}$ far in X. Then $A \notin \mathcal{G}$ so $\mathcal{G} \notin Cl_{X*}\, e_X A$ and thus $g(\mathcal{G}) \notin Cl_{Y*}\, e_Y B$ which implies that $B \notin g(\mathcal{G})$. As is easily shown, $f\mathcal{G} \subset g(\mathcal{G})$ and, since $f\mathcal{G}$ is both near and micromeric in Y

$$g(\mathcal{G}) = \{E \subset Y \mid \{E\} \cup f\mathcal{G} \text{ is near in } Y\}.$$

So, $\{B\} \cup f\mathcal{G}$ is far in Y.

(2) $\longrightarrow$ (3). Let $\mathcal{L}$ be far in Y and choose $\mathcal{A}$ as in (2). Let $A \in \mathcal{A}$ and let $B \in \mathcal{L}$ so that the condition in (2) holds. Let $\mathcal{G}$ be both near and micromeric in X with $\{A\} \cup$ $\cup\, \mathcal{G}$ far in X. Let

$$\mathcal{H} = \{H \subset X \mid \{H\} \cup \mathcal{G} \text{ is near in } X\}.$$

Then $\mathcal{H}$ is a cluster in X and $\{A\} \cup \mathcal{H}$ is far in X. So, $\{B\} \cup f\mathcal{H}$ is far in Y. Let

$$\mathcal{D} = \{D \subset Y \mid \{D\} \cup f\mathcal{G} \text{ is near in } Y\}$$

and

$$\mathcal{U} = \{U \subset Y \mid \{U\} \cup f\mathcal{H} \text{ is near in } Y\}.$$

Since $f\mathcal{G} \subset f\mathcal{H}$ and both of these are both near and micromeric in Y, then $\mathcal{D} = \mathcal{U}$. So, $\{B\} \cup f\mathcal{G}$ is far in Y.

(3) $\longrightarrow$ (2) is obvious because every cluster is both near and micromeric.

(2) $\longrightarrow$ (1). Let $g: X^* \longrightarrow Y^*$ be defined by

$$g(\mathcal{G}) = \{B \subset Y \mid \{B\} \cup f\mathcal{G} \text{ is near in } Y\}$$

for each $\mathcal{G} \in X^*$. Since $f\mathcal{G}$ is near and micromeric in Y, $g(\mathcal{G})$ is a cluster in Y. If $x \in X$, then the continuity of $f: X \longrightarrow Y$ guarantees that $fe_X(x) \subset e_Y(f(x))$ and so $g(e_X(x)) = e_Y(f(x))$. Therefore $g \circ e_X = e_Y \circ f$. To show that $g: X^* \longrightarrow Y^*$ is uniformly continuous, let $\mathcal{D}$ be far in Y^* and let

$$\mathcal{B} = \{B \subset Y \mid \text{for some } D \in \mathcal{D},\ D \subset \mathrm{Cl}_{Y^*} e_Y B\}.$$

Then $\mathcal{B}$ is far in Y so there exists $\mathcal{A}$ far in X with the conditions in (2) satisfied. It is sufficient to show that $\mathrm{Cl}_{X^*} e_X \mathcal{A}$ corefines $g^{-1}\mathcal{D}$. To that end, let $A \in \mathcal{A}$ and choose $B \in \mathcal{B}$ so that the condition in (2) is satisfied. There exists $D \in \mathcal{D}$ with $D \subset \mathrm{Cl}_{Y^*} e_Y B$. We shall show that $g^{-1}D \subset \mathrm{Cl}_{X^*} e_X A$. So, let $\mathcal{G} \in g^{-1}D$. Then $B \in g(\mathcal{G})$ which implies that $\{B\} \cup f\mathcal{G}$ is near in Y. The condition in (2) then implies that $\{A\} \cup \mathcal{G}$ is near in X. So, $A \in \mathcal{G}$ and thus $\mathcal{G} \in \mathrm{Cl}_{X^*} e_X A$.

Harris [12] defined a concept which he called WO-map in order to make the Wallman compactification a functor.

Bentley and Naimpally [5] generalized Harris' concept in order to make the Wallman-type compactifications of Steiner [28] a functor. The characterization of extendible maps given in the preceding proposition, although not a generalization of the WO-map concept, is a generalization of a slight variation of the WO-map concept. Maps satisfying condition (2) of Proposition 3.3 were investigated by Bentley [3].

3.4 Proposition. Let $f:X \longrightarrow Y$ be a uniformly continuous map between separated nearness spaces X and Y. Then

(1) If X is complete then $f:X \longrightarrow Y$ is extendible.

(2) If Y is regular then $f:X \longrightarrow Y$ is extendible.

(3) If X is a product and $f:X \longrightarrow Y$ is a canonical projection then $f:X \longrightarrow Y$ is extendible.

Proof: (1) is obvious and (2) is known (see Morita [23] and Herrlich [16]). In order to prove (3), let X_1 and X_2 be separated nearness spaces and let

$$X_1 \xleftarrow[p_1]{} X_1 \times X_2 \xrightarrow[p_2]{} X_2$$

be their product in <u>Near</u> with $f = p_1$. (This is general enough since we can write $X_2 = \prod_{j \neq i_0} X_j$.) We shall show that $f:X_1 \times X_2 \longrightarrow X_1$ is extendible by showing that (2) of Proposition 3.3 is true. Let $\mathcal{B}$ be far in X_1 and let $\mathcal{A} = f^{-1}\mathcal{B}$. For each $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ with $A = f^{-1}B$. If $B = \phi$ then we are through. So, suppose that $B \neq \phi$. Let $\mathcal{G}$ be a cluster in $X_1 \times X_2$ with $\{A\} \cup \mathcal{G}$ far in X. Then there exist $\mathcal{H}_1$, $\mathcal{H}_2$ far in X_1, X_2 respectively for which [x)]

$$f^{-1}\mathcal{H}_1 \vee p_2^{-1}\mathcal{H}_2 \text{ corefines } \{A\} \cup \mathcal{G} .$$

Since $\mathcal{G}$ is near in $X_1 \times X_2$ then

$$f^{-1}\mathcal{H}_1 \vee p_2^{-1}\mathcal{H}_2 \not\subset \mathcal{G} .$$

Consequently, there exist $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$ with

$$f_1^{-1} H_1 \cup p_2^{-1} H_2 \notin \mathcal{G} .$$

Therefore $A \subset f^{-1}H_1 \cup p_2^{-1}H_2$ and, since $A = f^{-1}B \neq \phi$ and $f^{-1}H_1 \cup p_2^{-1}H_2 \neq X$, then we must have $B \subset H_1$. We shall complete the proof by showing that

$$\mathcal{H}_1 \text{ corefines } \{B\} \cup f\,\mathcal{G} .$$

To that end, let $E \in \mathcal{H}_1$ and suppose that E contains neither B nor a member of $f\mathcal{G}$. Then

$$A \subset f^{-1}E \cup p_2^{-1}H_2 \text{ or } f^{-1}E \subset p_2^{-1}H_2 \in \mathcal{G} .$$

Since $B \not\subset E$, it must be that $f^{-1}E \cup p_2^{-1}H_2 \in \mathcal{G}$ and, since $\mathcal{G}$ is a grill, $f^{-1}E \in \mathcal{G}$ or $p_2^{-1}H_2 \in \mathcal{G}$. $f_1^{-1}H_1 \cup p_2^{-1}H_2 \notin \mathcal{G}$ implies that $p_2^{-1}H_2 \notin \mathcal{G}$, so it must be that $f^{-1}E \in \mathcal{G}$. But then $E = ff^{-1}E \in f\mathcal{G}$ which is a contradiction.

3.5 Proposition. The category $\underline{\text{SepNear}_e}$ is productive in $\underline{\text{SepNear}}$, where $\underline{\text{SepNear}_e}$ denotes the object full subcategory of $\underline{\text{SepNear}}$ whose morphisms are the extendible maps.

The Proposition 3.5 follows immediately from 3.4 and the following Lemma.

----

x) If $\mathcal{A}_1$ and $\mathcal{A}_2$ are collections of subsets of a space, then we write $\mathcal{A}_1 \vee \mathcal{A}_2 = \{A_1 \cup A_2 \mid A_1 \in \mathcal{A}_1 \text{ and } A_2 \in \mathcal{A}_2\}$.

Lemma: Let $(g_i : Y \longrightarrow Z_i)_{i \in I}$ be initial in <u>SepNear</u> and suppose that $f : X \longrightarrow Y$ is uniformly continuous on the separated nearness space X and that for all $i \in I$, $g_i \circ f : X \longrightarrow Z_i$ is extendible. Then $f : X \longrightarrow Y$ is extendible.

Proof: We apply Proposition 3.3. Let $\mathcal{B}$ be far in Y. Then for some finite subset J of I and for some family $(\mathcal{H}_j)_{j \in J}$ with each $\mathcal{H}_j$ far in Z_j, we have x)

$$\bigvee_{j \in J} g_j^{-1} \mathcal{H}_j \text{ corefines } \mathcal{B} .$$

For each $j \in J$ there exists $\mathcal{A}_j$ far in X such that for each $A \in \mathcal{A}_j$ there is $H \in \mathcal{H}_j$ for which whenever $\mathcal{C}$ is a cluster in X with $\{A\} \cup \mathcal{C}$ far in X then $\{H\} \cup (g_j \circ f)\mathcal{C}$ is far in Z_j. Let $\mathcal{A} = \bigvee_{j \in J} \mathcal{A}_j$. Then $\mathcal{A}$ is far in X. Let $A \in \mathcal{A}$. For some family $(A_j)_{j \in J}$ with each $A_j \in \mathcal{A}_j$, $A = \bigcup_{j \in J} A_j$. There exists a family $(H_j)_{j \in J}$ with each $H_j \in \mathcal{H}_j$ such that whenever $\mathcal{C}$ is a cluster in X with $\{A_j\} \cup \mathcal{C}$ far in X then $\{H_j\} \cup (g_j \circ f)\mathcal{C}$ is far in Z_j. There is $B \in \mathcal{B}$ with

$$B \subset \bigcup_{j \in J} g_j^{-1} H_j .$$

Let $\mathcal{C}$ be a cluster in X with $\{A\} \cup \mathcal{C}$ far in X. We must show that $\{B\} \cup f\mathcal{C}$ is far in Y. For all $j \in J$, $\{A_j\} \cup \mathcal{C}$ is far in X so $\{H_j\} \cup (g_j \circ f)\mathcal{C}$ is far in Z_j and $\{g_j^{-1} H_j\} \cup g_j^{-1} g_j f \mathcal{C}$ is far in Y. Let $\mathcal{D} = \bigvee_{j \in J} (\{g_1^{-1} H_j\} \cup g_j^{-1} g_j f \mathcal{C})$.

----

x) If $(\mathcal{A}_j)_{j \in J}$ is a family of collections of subsets of X then $\bigvee_{j \in J} \mathcal{A}_j = \{\bigcup_{j \in J} A_j \mid A_j \in \mathcal{A}_j\}$.

Then $\mathcal{V}$ is far in Y, and since $\mathcal{V}$ corefines $\{B\} \cup f\mathcal{G}$, we are through.

3.6 Example. $\underline{\text{SepNear}}_e$ does not have equalizers.

Proof: Let

$$A = \{(\tfrac{1}{n}, y) \mid n \in N \text{ and } -1 \leq y \leq 1\},$$

$$B = \{(x,0) \mid 0 \leq x \leq 1\}$$

and

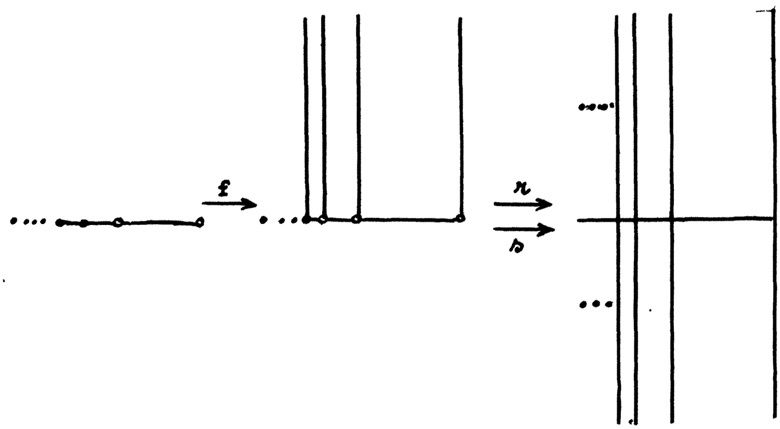
$$C = \{(\tfrac{1}{n}, y) \mid n \in N \text{ and } 0 \leq y \leq 1\}.$$

Let Z be the topological space on $A \cup B$ whose topology is generated by the usual topology plus A as a closed set. Let Y be the nearness subspace of Z on $B \cup C$. Let X be the nearness subspace of Y on $Y - (B \cap C)$, and let W be the nearness subspace of X on $X \cap B$.

Then the embedding $W \longrightarrow [0,1]$, $(x,0) \mapsto x$, and the inclusion $X \longrightarrow Y$ are the strict completions of W and X respectively (here $[0,1]$ is taken with the usual topology). The inclusion $f: W \longrightarrow X$ is a closed embedding but is not extendible (its unique pointwise continuous extension is not continuous at 0). Let $r,s: X \longrightarrow Z$ be defined by taking $r:$ $: X \longrightarrow Z$ to be the inclusion and $s: X \longrightarrow Z$ to be

$$s(x,y) = (x,-y).$$

Then r and s are extendible. Since f, being obviously the only candidate for an equalizer of r and s in $\underline{\text{SepNear}}_e$, is not extendible, the pair (r,s) has no equalizer in $\underline{\text{SepNear}}_e$. The following diagram illustrates the situation.



4. <u>Applications to H-closed extensions</u>. Harris [11] calls an open cover $\mathcal{G}$ of a topological space X a <u>p-cover of</u> X provided that the union of some finite subcollection of $\mathcal{G}$ is dense in X.

4.1 <u>Definition</u>. Associated with any Hausdorff topological space X is a nearness space HX on the same underlying set as X and with its nearness structure defined by: $\mathcal{A}$ is a uniform cover of HX iff $\mathcal{A}$ is refined by some p-cover of X.

The nearness collections in HX are characterized by: $\mathcal{A}$ is near in HX iff $\mathcal{A}$ has an adherence point in X or there exists a maximal open filter on X each member of which meets every member of $\mathcal{A}$.

The micromeric collections in HX are characterized by:

$\mathfrak{A}$ is micromeric in HX iff $\mathfrak{A}$ converges in X or there exists a maximal open filter on X which corefines $\mathfrak{A}$.

Harris [11] calls a map $f:X \longrightarrow Y$ between topological spaces a p-map iff the inverse image of a p-cover of Y is a p-cover of X. This relates to our terminology as follows: For a continuous map $f:X \longrightarrow Y$ between Hausdorff topological spaces, $f:X \longrightarrow Y$ is a p-map iff $f:HX \longrightarrow HY$ is uniformly continuous. Also according to Harris, a p-filter on a topological space X is an open filter on X which is either the open neighborhood system of a point or a maximal open filter that does not converge.

Our preceding results immediately imply the following three propositions.

4.2 Proposition. Let X be a Hausdorff topological space. Then

(1) HX is a separated nearness space and X is the underlying topological space of HX.

(2) The simple completion $\widetilde{HX}$ of HX is the Katětov extension of X.

(3) The strict completion $(HX)^*$ of HX is the Fomin extension of X.

4.3 Proposition (Harris [11]): Let $f:X \longrightarrow Y$ be a continuous map between Hausdorff topological spaces. Then f is extendible to the Katětov extensions of X and Y iff $f:X \longrightarrow Y$ is a p-map.

4.4 Proposition. Let $f:X \longrightarrow Y$ be a continuous map between Hausdorff topological spaces. Then $f:X \longrightarrow Y$ is extendible to the Fomin extensions of X and Y iff $f:X \longrightarrow Y$ is a p-map and the following condition is satisfied: If $\mathcal{H}$ is a p-cover of Y then there is a p-cover $\mathcal{G}$ of X such that for each $G \in \mathcal{G}$ there exists $H \in \mathcal{H}$ for which whenever $\mathcal{U}$ and $\mathcal{V}$ are p-filters on X and Y respectively with $G \in \mathcal{U}$ and $H \notin \mathcal{V}$ then there exist $U \in \mathcal{U}$ and $V \in \mathcal{V}$ with $V \cap fU = \phi$.

Proof: Use (1) $\longleftrightarrow$ (3) of Proposition 3.3.

Remark: We have heard recently that a result similar to 4.4 was known to D. Harris. As far as we know, this result is unpublished until now.

References

[1] B. BANASCHEWSKI: On the Katětov and Stone-Čech extensions, Canad. Math. Bull. 2(1959), 1-4.

[2] B. BANASCHEWSKI: Extensions of topological spaces, Canad. Math. Bull. 7(1965), 1-22.

[3] H.L. BENTLEY: Extensions of maps on nearness spaces, preprint.

[4] H.L. BENTLEY and H. HERRLICH: Extensions of topological spaces, in: Topology: Proc. Memphis State Univ. Conf. edited by S.P. Franklin and B.V. Smith Thomas (Marcel Dekker, New York, 1976), 129-184.

[5] H.L. BENTLEY and S.A. NAIMPALLY: Wallman T_1-compactification as epireflections, Gen. Top. Appl. 4 (1974), 29-41.

[6] J.W. CARLSON: B-completeness in nearness spaces, Gen. Top. Appl. 5(1975), 263-268.

[7] J.W. CARLSON: An ultrafilter completion for a nearness space, preprint.

[8] J.W. CARLSON: H-closed and countably compact exten-

sions, preprint.

[9] J. FLACHSMEYER: Zur Theorie der H-abgeschlossenen Erweiterungen, Math. Z. 94(1966), 349-381.

[10] S. FOMIN: Extensions of topological spaces, Ann. of Math. 44(1943), 471-480.

[11] D. HARRIS: Katětov extension as a functor, Math. Ann. 193(1971), 171-175.

[12] D. HARRIS: The Wallman compactification as a functor, Gen. Top. Appl. 1(1971), 273-281.

[13] N.C. HELDERMANN: Hüllenbildung in Nearness-Kategorien, Thesis, Freie Universität Berlin, 1977.

[14] N.C. HELDERMANN: On Hausdorff nearness spaces, preprint.

[15] H. HERRLICH: Topologische Reflexionen und Coreflexionen, Lect. Notes in Math. 78, Springer-Verlag, Berlin 1968.

[16] H. HERRLICH: A concept of nearness, Gen. Top. Appl. 4 (1974), 191-212.

[17] H. HERRLICH: Topological structures, Math. Centre Tracts 52, Amsterdam (1974), 59-122.

[18] H. HERRLICH and G. STRECKER: H-closed spaces and reflective subcategories, Math. Ann. 177(1968), 302-309.

[19] W. HUNSAKER and S.A. NAIMPALLY: Extensions of continuous functions; reflective functors, preprint.

[20] M. KATĚTOV: Über H-abgeschlossene und bikompakte Räume, Časopis Pěst. Mat. 69(1940), 36-49.

[21] M. KATĚTOV: On continuity structures and spaces of mappings, Comment. Math. Univ. Carolinae 6(1965), 257-278.

[22] J. MIODUSZEWSKI and L. RUDOLF: H-closed and extremally disconnected Hausdorff spaces, Rozprawy Mat. 66(1969), 1-52.

[23] K. MORITA: On the simple extension of a space with

respect to a uniformity, I-IV, Proc. Japan Acad. 27(1951), 65-72, 130-137, 166-171, 632-636.

[24] J. PORTER: Extension function and subcategories of Haus, Canad. Math. Bull. 18(1975), 587-590.

[25] J. PORTER and J. THOMAS: On H-closed and minimal Hausdorff spaces, Trans. Am. Math. Soc. 138 (1969), 159-170.

[26] J. PORTER and C. VOTAW: H-closed extensions I, Gen. Top. Appl. 3(1973), 211-224.

[27] J. PORTER and C. VOTAW: H-closed extensions II, preprint.

[28] E.F. STEINER: Wallman spaces and compactifications, Fund. Math. 61(1968), 295-304.

[29] A.K. STEINER and E.F. STEINER: On semi-uniformities, Fund. Math. 83(1973), 47-58.

University of Toledo x)
Toledo, Ohio 43606
U.S.A.

Universität Bremen
Bremen
Fed. Rep. Germany

(Oblatum 28.4. 1978)

----

x) The first named author was supported by a research grant from the Graduate School of the University of Toledo.