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IDEALS IN TRANSFORMATION SEMIGROUPS

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To SANDRA PEARCE with ever-lasting affection

Abstract: We first survey the work done on determining the one and two-sided ideals of various transformation semigroups, and then generalize this to cotransitive semigroups and the semigroup of transformations which shift less than ξ elements. In both cases, the completely semiprime ideals are characterized.

Key words: Ideal, transformation semigroup, shift, completely semiprime, reflective.

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1. Introduction. A number of authors have determined the ideals of certain transformation semigroups. For example, in [6] Malcev shows that every ideal of $\mathcal{T}_X$ has the form $T_\xi = \{\alpha \in \mathcal{T}_X : \text{rank}\ \alpha < \xi\}$ where $1 < \xi \le |X|'$ and in [16] Vorobev finds all ideals of $H(X, \aleph_0)$ (the subsemigroup of $\mathcal{T}_X$ shifting at most a finite number of elements in X) to be of the form $H_n = \{\alpha \in \mathcal{T}_X : \text{def}\,\alpha \ge n\}$ where $0 \le n < \aleph_0$. Next Liber [3] obtained a result for $\mathcal{S}_X$, and Sutov [12] one for $\mathcal{P}_X$, that is analogous to that of Malcev. Subsequently Sutov [11] also described the ideals of $W(X, \aleph_0)$ (see [10] for notation) in the following terms.

We say that $\alpha \in \mathcal{P}_X$ <u>strictly fixes</u> $x \in X$ if $x \in \operatorname{dom} \alpha$, $x\alpha = x$ and if $y\alpha = x$ then $y = x$. Let $Y \subseteq X$ and for $\alpha \in \mathcal{P}_X$, let $Y(\alpha)$ denote the elements of Y strictly fixed by α. Put $\Gamma(Y,\alpha) = Y \setminus Y(\alpha)$ and $\Gamma(\alpha, Y) = \operatorname{ran} \alpha \setminus Y(\alpha)$ and for each finite n satisfying $0 \leq n \leq |X \setminus Y|$, let $W(y,n)$ denote the set of all $\alpha \in W(X, \aleph_0)$ such that (i) all elements of $\operatorname{dom} \alpha$, except for at most a finite number of them, are contained in Y, and (ii) if $\Gamma(Y,\alpha)$ and $\Gamma(\alpha,Y)$ have cardinal p, q respectively, then $q \leq p + n$. Then every ideal of $W(X, \aleph_0)$ is a union of a family of sets of the form $W(Y,n)$. Finally in [13, 14] Sutov shows that the ideals of R_ι (the $\mathcal{R}$-class of $\mathcal{J}_X$ containing ι_X: that is, the semigroup of all one-to-one mappings from X to itself) can be identified with the sets $R(\xi) = \{\alpha \in R_\iota : \operatorname{def} \alpha \geq \xi\}$ for some $\xi \leq |X|$.

In § 2 of this paper we shall present a unification of some of the preceding work and in § 3 we shall generalize and simplify Sutov's results on $W(X, \aleph_0)$. In § 4 we itemize various types of ideals that have been used in the (abstract) theory of semigroups and interpret such concepts in the theory of transformation semigroups. Whenever possible we shall make special mention of one-sided ideals in transformation semigroups.

We are indebted to Professor G.B. Preston for stimulating the idea of this paper via a course of lectures in 1965 at Monash University: these were in turn founded on the seminar material summarized in [8].

2. Ideals determined by cardinals alone. We shall use the notation of [1] and [10]. In particular we recall that "transformation semigroup" connotes any subsemigroup of $\mathcal{P}_X$ where X is an arbitrary non-empty set; that ξ' denotes the successor of the cardinal ξ ; and that $\{x_i\}$ will, within context, signify a set of elements x_i indexed by some (unspecified) set I (see [1], Vol. 2, p. 241). We shall also adopt the standard convention of writing $\alpha \in \mathcal{P}_X$ as

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

where $X\alpha = \{x_i\}$, $X/\alpha \circ \alpha^{-1} = \{A_i\}$ and $A_i = x_i\alpha^{-1}$. Note that this also means $\operatorname{dom} \alpha = \cup A_i$. Finally if S is any transformation semigroup, we write

$$S_\xi = \{\alpha \in S : \operatorname{rank} \alpha < \xi\}$$

which, in the notation of [10], equals $S \cap P_\xi$.

To unify some of the work summarized in § 1, we now define a transformation semigroup S to be cotransitive if whenever $\alpha \in S$ and

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

then (i) for each $\{y_i\} \subseteq X$, there exists $\lambda \in S$ such that $y_i \in x_i\lambda^{-1}$, and (ii) for each $\{b_i\} \subseteq X$, there exists $\mu \in$ $\in$ S such that

$$\mu = \begin{pmatrix} A_i \\ b_i \end{pmatrix}$$

<u>Theorem 1</u>. If S is any cotransitive subsemigroup of $\mathcal{P}_X$, then the non-zero ideals of S are precisely the sets S_ξ where $1 < \xi \leq |X|'$. In particular, the principal ideals of S are of the form $S_{\xi'}$ where $\xi \geq 1$.

Proof. Since rank $(\alpha\beta) \leq$ min (rank α, rank β), each S_ξ is an ideal. Conversely suppose I is an ideal of S and let ξ equal the least cardinal greater than rank β ($\beta \in I$). Then $I \subseteq S_\xi$. Let $\alpha \in S_\xi$. If rank $\beta <$ rank α for all $\beta \in I$, we have a contradiction. Hence there exists $\beta \in I$ with rank $\beta \geq$ rank α. Let

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} B_j \\ y_j \end{pmatrix},$$

and choose a partial cross-section $\{b_i\}$ of $\{B_j\}$. Then since S is cotransitive, there exist λ, $\mu \in S$ such that

$$\lambda = \begin{pmatrix} A_i \\ b_i \end{pmatrix}$$

and $b_i\beta \in x_i\mu^{-1}$, so that $\alpha = \lambda\beta\mu \in I$ and we have I = $= S_\xi$ as required.

Remark 1. It is easy to see that each of $\mathcal{G}_X$, $\mathcal{T}_X$ and $\mathcal{P}_X$ are cotransitive and so the results of Malcev [6], Liber [3] and Sutov [12] are immediate consequences of Theorem 1. The reference to principal ideals generalizes an observation in [8].

A description of the principal one-sided ideals of any cotransitive semigroup is contained in the following result: it unifies the characterizations of Green's $\mathcal{L}$ and $\mathcal{R}$ relations on $\mathcal{T}_X$ [1, Vol. 1], $\mathcal{I}_X$ [9] and $\mathcal{P}_X$ [2]; we omit the (easy) proof.

Theorem 2. If S is a cotransitive semigroup and α, $\beta \in S$, then

(i) $\alpha \in S^1.\beta$ if and only if $X\alpha \subseteq X\beta$, and

(ii) $\alpha \in \beta.S^1$ if and only if $\beta \circ \beta^{-1} \subseteq \alpha \circ \alpha^{-1}$.

Since any left ideal of a semigroup is a union of principal left ideals, we have in fact described all left ideals of any cotransitive semigroup. In [13, 14] Sutov showed that the Baer-Levi semigroup $M_{|X|}$ (see [1], Vol. 2) has for its left ideals the subsets N of $M_{|X|}$ with the property: if $\beta \in N$ and $X\alpha \subseteq X\beta$ for some $\alpha \in M_{|X|}$ where $|X\beta \setminus X\alpha| = |X|$, then $\alpha \in N$. We generalize this to any Baer-Levi semigroup M_q by showing:

Theorem 3. If $\alpha, \beta \in M_q$, $\aleph_0 \leq q \leq |X|$, then $\alpha = \lambda\beta$ for some $\lambda \in M_q$ if and only if $X\alpha \subseteq X\beta$ and $|X\beta \setminus X\alpha| = q$.

Proof. If $X\alpha \subseteq X\beta$ then $\alpha = \lambda\beta$ for some λ such that $\lambda \circ \lambda^{-1} = \alpha \circ \alpha^{-1}$. Hence λ is one-to-one and we have

$$|X \setminus X\lambda| = |(X \setminus X\lambda)\beta| = |X\beta \setminus X\lambda\beta| = q,$$

as required for $\lambda \in M_q$. The converse is equally clear.

It is well known that the Baer-Levi semigroups are right simple (see [1], Vol. 2, Theorem 8.2). This is not true of the partial Baer-Levi semigroups $N_q = \{\alpha \in \mathcal{I}_X : \operatorname{def}\alpha = q\}$.

Theorem 4. If $\alpha, \beta \in N_q$, $\aleph_0 \leq q \leq |X|$, then $\alpha = \beta\lambda$ for some $\lambda \in N_q$ if and only if $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$.

Proof. If $\alpha = \beta\lambda$ then $\beta \circ \beta^{-1} \subseteq \alpha \circ \alpha^{-1}$ which in turn is true if and only if $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$. Hence if $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$, then there exists $\lambda \in \mathcal{P}_X$ such that $\alpha = \beta\lambda$ where in fact λ can be chosen so that $\lambda \in \mathcal{I}_X$ and $X\lambda = X\alpha$.

To close this section we note that Sutov [13, 14] has also shown that every left ideal of R_L is of the form $\{\alpha \in R_L : X\alpha \subseteq Y\}$ for some fixed $Y \subseteq X$; that R_L is right simple is clear from its definition.

3. One and two sided ideals of $W(X, \xi)$. Throughout this section X will be an infinite set and ξ an infinite cardinal. As in [10] we let $W(X, \xi)$ (or W for short) denote the semigroup $\{\alpha \in \mathcal{P}_X : |S(\alpha)| < \xi\}$ where $S(\alpha) = \{x \in \operatorname{dom}\alpha : x\alpha \neq x\}$. Clearly, $W(X, \xi) = \mathcal{P}_X$ when $\xi = |X|'$. Sutov [11] has observed that each $W(Y,n)$ is an ideal of $W(X, \aleph_0)$ and that in particular when $|Y| = m < \aleph_0$, then $W(Y,n) = \{\alpha \in W(X, \aleph_0) : \operatorname{rank}\alpha \leq m + n\}$.

In this section we intend to characterize all the one and two sided ideals of $W(X, \xi)$. For this purpose we begin with

Lemma 1. If L, R are respectively left, right ideals of W then

(i) if $\alpha \in L$, then $\iota_{X\alpha} \in L$, and

(ii) if $\alpha \in R$, then $\delta\alpha = \alpha$ for some idempotent $\delta \in R \cap \mathcal{I}_{\mathrm{dom}\,\alpha}$.

Proof. We start by writing $\alpha \in W$ as

$$\alpha = \begin{pmatrix} A_i & u_j & y_k \\ x_i & v_j & y_k \end{pmatrix}$$

where $|A_i| > 1$ for each i (such $\alpha \circ \alpha^{-1}$ - classes may not exist, in which case the following argument still holds with suitable re-interpretation), $u_j \neq v_j$ for each j, and $|I \cup J| < \xi$. For each i, choose $a_i \in A_i \setminus x_i$ and define a mapping γ by

$$\gamma = \begin{pmatrix} x_i & v_j & y_k \\ a_i & u_j & y_k \end{pmatrix}.$$

Then γ is well defined since $\{x_i\}$, $\{v_j\}$, $\{y_k\}$ are pairwise disjoint, as are $\{a_i\}$, $\{u_j\}$ and $\{y_k\}$. Moreover $|S(\gamma)| < \xi$, $\gamma\alpha = \iota_{X\alpha}$, and $\alpha\gamma = \delta$ say, is an idempotent such that $\delta\alpha = \alpha$, and the result follows.

The next result now follows as an easy consequence.

Theorem 5. Let Σ be a family of subsets of X and put

$$L_\Sigma = \cup \{W \cdot \iota_A : A \in \Sigma\}, \quad I_\Sigma = \cup \{W \cdot \iota_A \cdot W : A \in \Sigma\}.$$

For each $A \in \Sigma$, choose an idempotent $\delta_A \in \mathcal{J}_A \cap W$ and put

$$R_{\Sigma} = \cup \{ \delta_A \cdot W : A \in \Sigma \} .$$

Then L_{Σ}, R_{Σ}, I_{Σ} are left, right, two-sided ideals of W and all ideals of each kind are obtained in this way.

Now let $\Sigma_{\xi} = \{ A \subseteq X : |A| < \xi \}$ and put $P = \mathcal{P}_X$. Note that $P_{\xi} = \cup \{ P \cdot \iota_A \cdot P : A \in \Sigma_{\xi} \}$. For, if $\alpha \in P_{\xi}$, then $\alpha = \alpha \cdot \iota_{X\alpha} \cdot \iota_{X\alpha}$ and $X\alpha \in \Sigma_{\xi}$, and conversely if $\alpha \in$ $\in P \cdot \iota_A \cdot P$ for some $A \in \Sigma_{\xi}$, then rank $\alpha < \xi$ (since rank $(\alpha\beta) \leq$ rank α, rank β). We also have

Corollary 1. Let Σ be a family of subsets of X and for each $A \in \Sigma$, choose an idempotent $\delta_A \in \mathcal{J}_A$. Then

$$L_{\Sigma} = \cup \{ P \cdot \iota_A : A \in \Sigma \}, \quad R_{\Sigma} = \cup \{ \delta_A \cdot P : A \in \Sigma \}$$

are respectively left, right ideals of $\mathcal{P}_X$ and every left, right ideal of $\mathcal{P}_X$ can be obtained in this way.

4. Completely semiprime ideals. We follow the terminology of [7] rather than [4] and call an ideal I of a semigroup S completely prime [completely semiprime] if $ab \in I$ implies $a \in I$ or $b \in I$ [$a^2 \in I$ implies $a \in I$]. In this section we shall show that for $\mathcal{J}_X$, $\mathcal{P}_X$ and $\mathcal{S}_X$, completely semiprime ideals (and so also completely prime ideals) are virtually non-existent, whereas $W(X, \xi)$, $\xi \leq |X|$, contains an infinite number.

Theorem 6. If $|X| = n < \aleph_0$ and S denotes any one of

$\mathcal{T}_X$, $\mathcal{I}_X$ or $\mathcal{P}_X$, then S_n is the only proper completely semiprime ideal of S.

Proof. We know that S_r, $1 < r \leq n$, are the proper non-zero ideals of S- Suppose $1 < r < n$ and choose $y_1, \ldots, y_{r-1}$ in X, $u \in Y \setminus \{y_1, \ldots, y_{r-1}\}$ (denoted by Y say), and $v \in Y \setminus u$ (non-empty since $r < n$). Now if S equals $\mathcal{T}_X$ or $\mathcal{P}_X$, define $\lambda \in \mathcal{T}_X$ by

$$\lambda = \begin{pmatrix} y_1 & y_2 & \cdots & y_{r-1} & Y \\ v & y_2 & \cdots & y_{r-1} & u \end{pmatrix}$$

and if S equals $\mathcal{I}_X$, define $\lambda \in \mathcal{I}_X$ by

$$\lambda = \begin{pmatrix} y_1 & \cdots & y_{r-1} & u \\ y_1 & \cdots & y_{r-1} & v \end{pmatrix}$$

Then in all three cases, $\lambda^2 \in S_r$, but $\lambda \notin S_r$. Finally if $\lambda^2 \in S_n$, but $\lambda \notin S_n$, then $\lambda \in \mathcal{G}_X$ and so $\lambda^2 \in \mathcal{G}_X$, a contradiction.

<u>Theorem 7.</u> If X is infinite, then none of $\mathcal{T}_X$, $\mathcal{I}_X$, $\mathcal{P}_X$ contains a proper completely semiprime ideal.

Proof. We know that if S denotes any one of $\mathcal{T}_X$, $\mathcal{I}_X$ or $\mathcal{P}_X$, then S_ξ $(1 < \xi \leq |X|)$ are the proper non-zero ideals of S. The following argument implies whether ξ is finite or not. Choose $a \in X$ and select a partition $\{\{x_i\}, \{y_i\}\}$ of $X \setminus a$ in which $|I| \geq \xi$. Now define λ_1 by

$$\lambda_1 = \begin{pmatrix} y_i & X \setminus \{y_i\} \\ x_i & a \end{pmatrix}$$

and put $\lambda_2 = \lambda_1 | \{y_i\}$. Then in the appropriate cases we have $\lambda_i^2 \in S_\xi$ but $\lambda_i \notin S_\xi$, and thus no S_ξ is completely semiprime.

To determine the proper completely semiprime one-sided ideals of $W(X, \xi)$, we first recall that if $\Sigma \subseteq \mathcal{P}(X)$, then $A \in \Sigma$ is <u>maximal in</u> Σ if $A \subseteq B$ and $B \in \Sigma$ implies $A = B$.

<u>Theorem 8.</u> Let X be an arbitrary set and for each $a \in X$, let $A = X \setminus a$ and $\mathcal{A} = \{A : a \in X\}$. If $\Sigma \subseteq \mathcal{P}(X)$ and contains a maximal element, then L_Σ is a proper completely semiprime left ideal of W_ξ $(\xi \geq \aleph_0)$ if and only if $X \notin \Sigma$ and $A \in \Sigma$ for each $A \in \mathcal{A}$ (in which case $L_\Sigma = \cup\{W \cdot \iota_A : A \in \mathcal{A}\}$).

Proof. If $L_\Sigma = W$, then $\iota_X = \alpha \cdot \iota_B$ for some $B \in \Sigma$ and $\alpha \in W$ so that $B = X$. If $X \in \Sigma$, then for all $\alpha \in W$, $\alpha = \alpha \cdot \iota_X \in L_\Sigma$. This establishes that L_Σ is proper if and only if $X \notin \Sigma$.

Now suppose L_Σ is proper and completely semiprime and let B be maximal in Σ . If $|X \setminus B| \geq 2$, we can choose $s, t \in X \setminus B$ and put

$$\lambda = \begin{pmatrix} s & b_i \\ t & b_i \end{pmatrix}$$

where $B = \{b_i\}$. Then $\lambda^2 = \iota_B \in L_\Sigma$ implies $\lambda \in L_\Sigma$, so that $\lambda = \alpha \cdot \iota_C$ for some $\alpha \in W$ and $C \in \Sigma$. Now $B \subseteq B \cup t \subseteq C$ implies (by maximality of B) that $t \in B$, a contradiction. Moreover $|X \setminus B| \neq 0$ by our initial remark.

Hence $|X \setminus B| = 1$ and $B = X \setminus a$ for some $a \in X$.

Now choose $b \in B$ and put $Y = B \setminus b = \{y_j\}$ say. If we write

$$\mu = \begin{pmatrix} b & y_j \\ a & y_j \end{pmatrix}$$

then $\mu^2 = \iota_Y = \iota_Y \cdot \iota_B \in L_\Sigma$ implies $\mu \in L_\Sigma$, and so $\mu = \omega \cdot \iota_D$ for some $D \in \Sigma$ and $\omega \in W$. Then $X \setminus b = Y \cup a \subseteq$ $\subseteq D$. Since $D \neq X$, we obtain $D = X \setminus b$; that is, $\mathcal{A} \subseteq \Sigma$ as asserted.

For the converse suppose Σ does not contain X but does contain every $A \in \mathcal{A}$ and let $\alpha^2 = \omega \cdot \iota_B$ for some $\omega \in W$ and $B \in \Sigma$. Then α is not onto X and so there exists $a \in X$ such that $X\alpha \subseteq X \setminus a = A$. Hence $\alpha = \alpha \cdot \iota_A \in L_\Sigma$ and thus L_Σ is completely semiprime.

Theorem 9. With the same notation as in Theorem 8, we have: R_Σ is a proper completely semiprime right ideal of W_ξ if and only if $X \notin \Sigma$, $\mathcal{A} \subseteq \Sigma$ and for each $A \in$ $\in \mathcal{A}$, the idempotents $\sigma_A \in \mathcal{J}_A$ equal ι_A (in which case $R_\Sigma = \cup\{\iota_A \cdot W : A \in \mathcal{A}\}$).

Proof. Suppose R_Σ is proper and completely semiprime and that also $X \in \Sigma$. As an abbreviation, let $\sigma = \sigma_X$ and assume $X\sigma = \{a\}$. Then we must have $\xi = |X|'$ and for $b \in X \setminus a$, the mapping

$$\lambda = \begin{pmatrix} X \setminus \{a,b\} & \{a,b\} \\ a & b \end{pmatrix}$$

belongs to W. However $\alpha^2 = \sigma \cdot a_b \in R_{\Sigma}$, whereas $\alpha \notin R_{\Sigma}$ (since dom $\alpha = X$ and rank $\alpha = 2$). Hence rank $\sigma \geq 2$. Now since $\sigma \neq \iota_X$, we can choose $a \in X\sigma$ such that $|a\sigma^{-1}| \geq 2$. Put $B = a\sigma^{-1}$ and write

$$\sigma = \begin{pmatrix} B & B_1 & B_k \\ a & b_1 & b_k \end{pmatrix}, \qquad \lambda = \begin{pmatrix} B \setminus a & B_1 \cup a & B_k \\ a & b_1 & b_k \end{pmatrix}$$

where the index set K is possibly empty. Then

$$\lambda^2 = \begin{pmatrix} B \cup B_1 & B_k \\ b_1 & b_k \end{pmatrix} = \sigma \cdot \begin{pmatrix} \{a, b_1\} & b_k \\ b_1 & b_k \end{pmatrix} \in R_{\Sigma}$$

and so $\lambda \in R_{\Sigma}$. Since dom $\lambda = X$, we must therefore have $\lambda = \sigma\beta$ for some $\beta \in W$. However if $b \in B \setminus a$, then $a\beta = b\sigma\beta = b\lambda = a$, whereas $a\lambda = b_1 \neq a\sigma\beta$. Hence if R_{Σ} is proper and completely semiprime then $X \notin \Sigma$.

We now proceed as in the proof of Theorem 8 by also letting B be a maximal element of Σ : if $|X \setminus B| \geq 2$, we choose $s, t \in X \setminus B$ and let λ be the mapping with domain $B \cup s$ such that $s\lambda = t$ and $\lambda | B = \sigma_B$. Then $\lambda \in R_{\Sigma}$ and so $\lambda = \sigma_C \cdot \omega$ for some $C \in \Sigma$ and $\omega \in W$; this implies $\sigma_B \cdot \lambda = \lambda$, so that $B \subseteq \operatorname{dom} \lambda \subseteq C$ and we find that $B = X \setminus a$ for some $a \in X$. To show $\mathcal{A} \subseteq \Sigma$, choose $b \neq a$ and write

$$\sigma_B = \begin{pmatrix} D & D_i \\ y & a_i \end{pmatrix}, \quad \mu = \begin{pmatrix} (D \setminus b) \cup a & D_i \\ b & d_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} D_i \\ d_i \end{pmatrix}$$

where $b \in D$. Then $\mu^2 = \gamma = \delta_B \cdot \gamma \in R_\Sigma$, and so $\mu = \delta_E \cdot \beta$ for some $E \in \Sigma$ and $\beta \in W$. It now follows that

$$X \setminus b = (B \setminus D) \cup (D \setminus b) \cup a = \operatorname{dom} \mu \subseteq E$$

and thus $E = X \setminus b$. Moreover if $A \in \mathcal{A}$ and $\delta_A \neq \iota_A$, we have a contradiction similar to that indicated in the preceding paragraph.

For the converse we simply note that if $R_\Sigma = W$ then $\iota_X = \iota_C \cdot \alpha$ for some $C \in \Sigma$ and we obtain $C = X$; furthermore if $\lambda^2 = \delta_B \cdot \omega$ for some $B \in \Sigma$ and if $X \notin \Sigma$, then $\operatorname{dom} \lambda \neq X$ and so $\operatorname{dom} \lambda \subseteq A$ for some $A \in \mathcal{A}$. Hence if $\mathcal{A} \subseteq \Sigma$ and each δ_A equals ι_A $(A \in \mathcal{A})$, we have $\alpha = \iota_A \cdot \alpha \in R_\Sigma$ and the result follows.

<u>Corollary</u>. With the notation of Theorem 8 and if I denotes $\cup\{W \cdot \iota_A \cdot W : A \in \mathcal{A}\}$, then I is the only proper completely semiprime ideal of W. Moreover, if L_ι denotes the $\mathcal{L}$-class of $\mathcal{T}_X$ containing ι_X (that is, the set of all mappings from X onto itself) then $I = W \setminus L_\iota$.

To complete this section we briefly consider the existence of <u>reflective</u> ideals in transformation semigroups: that is, an ideal I such that $ab \in I$ implies $ba \in I$ [5, 15].

<u>Theorem 10.</u> If X is infinite, then none of $\mathcal{T}_X$, $\mathcal{S}_X$ or $\mathcal{P}_X$ contains a proper reflective ideal.

Proof. If T_ξ $(\xi \leq |X|)$ is proper and reflective, partition X into $A \cup B$ where $|A| = \xi$ and $|B| \geq \xi$, and fix some $b \in B$. Now let $C = \{e_i\}$ be any subset of $B \setminus b$ with cardinal ξ and put $A = \{a_i\}$. Finally write

$$\alpha = \begin{pmatrix} X\setminus A & a_i \\ b & a_i \end{pmatrix} \quad \text{and } \beta = \begin{pmatrix} X\setminus C & c_i \\ b & a_i \end{pmatrix}$$

Then rank $(\alpha\beta) = 1$ but rank $(\beta\alpha) = \xi + 1$. This proves the result for $\mathcal{T}_X$ and $\mathcal{P}_X$; the modification required for $\mathcal{I}_X$ is obvious.

Theorem 11. If $|X| = n < \aleph_0$ and S denotes any one of $\mathcal{T}_X$, $\mathcal{I}_X$ or $\mathcal{P}_X$, then S_n is the only proper reflective ideal in S.

Proof. If $\alpha, \beta \in S$ and rank $(\alpha\beta) < n$ when rank $(\beta\alpha) = n$, then both $\alpha, \beta \in \mathcal{G}_X$ and we have a contradiction. Hence S_n is reflective in each case.

To conclude the proof for $\mathcal{T}_X$ and $\mathcal{P}_X$, let $r < n - 1$ and note that if we write

$$\alpha = \begin{pmatrix} 1 & 2 \ldots r-1 & r & Y \\ 2 & 3 \ldots\ \ r & r+1 & r+2 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 2 & 3 \ldots\ \ r & 1 & Y \\ 1 & 2 \ldots r-1 & r & r+1 \end{pmatrix}$$

where $Y = X\setminus\{1,\ldots,r\}$, then rank $(\alpha\beta) = r$ whereas rank $(\beta\alpha) = r + 1$; obvious modifications to α and β also establish the result for $\mathcal{I}_X$.

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