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ON A WEAK KELLEY-MORSE THEORY OF CLASSES

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Abstract: We show that in the Kelley-Morse theory of classes without powerset axiom one can interpret the same theory together with the axiom of constructibility and some forms of (generalized) continuum hypothesis. An application to interpretability problems of the Alternative Set Theory of Vopěnka is shown, too.

Key words: Kelley-Morse theory, Alternative Set Theory, interpretation, continuum hypothesis, constructibility.

AMS: Primary 02K05, 02K10, 02K25

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§ 1. Preliminaries. In this paper we consider the Kelley-Morse theory of classes (cf. [3]) without powerset axiom (denotation KM_-). This theory is roughly speaking the impredicative extension of ZF^- (i.e. the Zermelo-Fraenkel set theory without the powerset axiom). Let us stress that in KM_- the choice scheme is not contained and moreover that neither the axiom of choice is used in the following construction. We are going to show that in this theory the notion of constructible class can be introduced. Instead, however, of repeating Gödel's proof (which is possible) we use instead of constructibility predicate (for classes) the ramified-analytical construction (see [3]). This corresponds to "strongly constructible" sets of Cohen (see [2]) and

Gandy-Putnam construction of the least β-model of second order arithmetic (see [1]). Similarly to the fact that the ramified analysis over $\langle \omega, +, \cdot, < \rangle$ satisfies "Every real is constructible" our construction yields, over each β-model of KM_- the least β-extension of its L-part (to see this analyse the proof of our interpretability result similarly as it is done in [3]). Our proof is heavily based on the methods of [3], Section 2.

The ramified analytical hierarchy over an arbitrary structure $\mathfrak{M}$ is defined as follows:

$R.A._0^{\mathfrak{M}} = |\mathfrak{M}| \cup$ "family of all subsets of $|\mathfrak{M}|$ parametrically definable in $\mathfrak{M}$ "

$R.A._{\alpha+1}^{\mathfrak{M}} = |\mathfrak{M}| \cup$ "family of all subsets of $|\mathfrak{M}|$ definable over $\langle \mathfrak{M}, R.A._{\alpha}^{\mathfrak{M}} \rangle$ "

$R.A._{\lambda}^{\mathfrak{M}} = \bigcup_{\xi < \alpha} R.A._{\xi}^{\mathfrak{M}}$ for limit λ

$R.A.^{\mathfrak{M}} = \bigcup_{\xi \in On} R.A._{\xi}^{\mathfrak{M}}$

This construction can be trivially generalized using arbitrary wellordering instead of the class of all ordinal numbers and we are going to use this generalization. Moreover our construction (admitting $\mathfrak{M}$ to be a proper class) may be formalized in a two sorted language and in particular in Kelley-Morse type theory of classes. If $\mathfrak{M}$ possesses a definable wellordering and an appropriate coding scheme then $R.A.^{\mathfrak{M}}$ has a definable wellordering. Elaboration of some of these facts is done in [3], Section 2.

We remind that W.O.(X) denotes that X is wellordering; if Φ is a formula and P a predicate then Φ^P is a restriction of Φ to P.

§ 2. Interpretation. We are going to construct an interpretation of $KM_- + V = L$ in KM_-. So from now on work in KM_-.

We have the following two cases (to start with):

(i) ω_1^L exists (i.e. it is a set)

Then the interpretation of $KM_- + V = L$ in KM_- is the following: Sets are interpreted as elements of $L_{\omega_1^L}$, classes are subsets of $L_{\omega_1^L}$ which are constructible. This interpretation forms a set or proper class depending on whether the ω_2^L is a set or is just On.

(ii) ω_1^L does not exist (i.e. it is not a set)

This is the principal case and the moment of reflection shows that On plays the role of ω_1^L. The rest of this section will be devoted to this case.

Now we start the construction of ramified analysis in the fashion described in [3], Section 2 using wellorderings of the class On and having in mind however that we do not have the powerset axiom here. Let T be a variable for (class) wellordering of On. In our interpretation the role of $\mathfrak{M}$ will play the class of all constructible sets and therefore the index $\mathfrak{M}$ will be omitted. Our definition of $R.A._T$ is rather informal and precise definition using the predicate U_T can be found in [3].

Note that since $\mathbb{L}$ has a definable wellordering, the prewellordering $\mathcal{V}_T$ of [3] is actually a wellordering of $R.A._T$.

We are going to consider two possible cases and treat them almost simultaneously, though the situation is quite different in each of them:

(a) <u>There is a wellordering</u> T <u>such that</u> $R.A._{T+1}$ <u>does not contain a wellordering of the type</u> $\geq T$.

(b) <u>The case (a) does not hold</u>.

If (a) holds then the desired interpretation is as follows: Sets are elements of L, classes are elements of $R.A._{T_0}$ (i.e. those classes which are sections of U_{T_0}) where T_0 is the shortest wellordering T such that $R.A._{T+1}$ does not contain a wellordering of the type $\geq T$.

If (b) holds then the desired interpretation is as follows: Sets are elements of L, classes are members of union (through all wellorderings in question) of $R.A._T$ (in fact those which are sections of U_T for any T).

Now we have to show the following three key facts:

I. There is no proper semiset in $R.A._{T_0}$ (R.A. resp.) i.e. if such a class is included in a set it is a set itself.

II. β-property of $R.A._{T_0}$ (R.A. respectively) i.e. if $W.O.^{R.A._{T_0}}(S)$ ($W.O.^{R.A.}(S)$ respectively) then $W.O.(S)$. Thus we show that $R.A._{T_0}$ (R.A. respectively) is a β-interpretation (see [8]).

III. Our interpretation interprets the comprehension scheme.

<u>Fact I</u>. In order to deal successfully with I we notice that since we have a definable wellordering of $R.A._{T_0}$ (R.A. respectively) we are able to imitate the Gödel's proof

for GCH while proceeding as follows: Take X a subclass of some L_α , α is denumerable in L since ω_1^L does not exist. Assume $X \in R.A._T$. By Skolem-Löwenheim argument we have an elementary denumerable $M \prec R.A._T$. By contraction we get denumerable $Z \subseteq P(L_\beta)$ for some $\beta > \alpha$ but $\beta <$ On. We get $\pi : M \simeq L_\beta \cup Z$. Now Z is $R.A._\nu^{L_\beta}$, ramified analytical level over L_β for some ν. But L is a ZF^- model so it is closed under R.A. construction. Thus X which is not moved under π belongs to $R.A._\nu^{L_\beta}$. Thus $Z \in L$.

The fact I takes also care of the lower case of continuum hypothesis; reals become wellordered in type On (which plays the role of ω_1^L) such that every proper segment is denumerable.

Fact II. Let $W.O.^{R.A._{T_0}}(X)$ ($W.O.^{R.A.}(X)$ respectively). Uniformly both for case (a) and (b) we are going to show W.O.(X). Indeed, if not, then there is a denumerable descending sequence Z for X. By the axiom of replacement there must be $\alpha <$ On such that $Z \subseteq \alpha$. But L is a ZF^- model, thus (by I) $X \cap \alpha^2 \in L$ is similar to an ordinal which is absurd.

Fact III. In both cases (a) and (b) we prove reflection principle for $R.A._{T_0}$ (R.A. respectively). In the proof the line of [3] is followed. (Since we do have the definable wellordering $<$ of $R.A._{T_0}$ (R.A. respectively) we do not need the selection principle of [3] . That principle took case of the case when we had only "good" prewellordering of all classes but used the powerset axiom). Here we proceed as follows (we point out only the main steps): If the case (a) holds then for every wellordering $S < T_0$ there is an isomorphic wellordering which is in $R.A._{T_0}$ (by the minimality of T_0). In case (b) we have

Lemma. Under the assumption of (b) for each wellordering S there is an R.A.-wellordering T such that $T \simeq S$.

Proof. Let us assume that our statement is false, i.e. that there is S which is not isomorphic to any R.A.-wellordering. Then S must be longer than all R.A.-wellorderings since the latter family is closed under initial segments. But then there is no wellordering of the type S in $R.A._{S+1}$ contradicting assumptions.

Having the lemma we proceed as follows:

In case (a) we first show that $R.A._{T_0}$ has enough closure properties (see [3], Section 2) and then if we could not find a bound for existencial quantifier (this is the only nontrivial case of reflection principle) then there would exist a wellordering of the type $\geq T_0$, definable over $R.A._{T_0}$.

In case (b) we simply can assume that all wellorderings under consideration are R.A. and having in mind that R.A. has a definable wellordering we proceed as in case (a).

Thus we completed the proof of the following theorem:

Theorem. (In KM_-.) There are definable predicates P (unary) and Q (binary) such that

(1) All axioms of KM_-, relativized to P, hold

(2) $(V = HC)^P$ holds

(3) $(V = L)^P$ holds

(4) W wellorders P in such a way that every initial

segment of Q is codable as a class.

Thus two instances of general continuum hypothesis corresponding to $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ hold in the interpretation.

We notice that the idea of considering an alternative (a)$\vee$(b) is due to Gandy (unpublished).

§ 3. AST <u>and</u> KM_- <u>are mutually interpretable</u>. P. Vopěnka built the alternative set theory as an alternative to the Cantor's set theory (see [6]). A formalization of Vopěnka's theory can be found in [4], here we are going to describe an equivalent axiomatic system (cf. [5]).

Let ZF_{Fin} denote the Zermelo-Fraenkel set theory (ZF) in which the axiom of infinity is replaced by its negation.

The Alternative set theory (AST) is the theory with the language consisting of one sort of (class) variables and two binary predicates $\in$ and $=$. Sets are defined as members of classes.

We define that a set x is finite if every its subclass is a set (in symbols $\mathrm{Fin}(x) \equiv (\forall X \subseteq x)\mathrm{Set}(X)$). $\preceq$ is called a wellordering if it is a linear ordering such that every non empty subclass of the field of $\preceq$ has the minimal element (i.e. $(\forall X)(0 \neq X \subseteq \mathrm{dom}(\preceq) \longrightarrow (\exists x \in X)(\forall y \in X)(\neg\, y < x))$). A class X is countable (in symbols Count(X)) if it is not finite and if there is a wellordering $\preceq$ of it such that $(\forall x)\ \mathrm{Fin}(\{y; y \preceq x\})$.

It is necessary to stress the fact that all sets in AST are finite from the Cantor's point of view (this is a consequence of the third axiom). On the other hand we admit proper classes which are subclasses of sets (the existence of such classes follows from the prolongation axiom) and hence there are sets x with $\neg \mathrm{Fin}(x)$. The class of all natural numbers is not wellordered by $\in$ since the class of all "non-standard" natural numbers (i.e. the class of all natural numbers which are not finite) has no minimal element (cf. [6] or [5]).

We accept the following axioms

1) Axiom of extensionality (for classes)

2) Scheme of existence of classes

For <u>every</u> formula $\varphi(X, X_1, \ldots, X_n)$ we accept the axiom

$$(\forall X_1)\ldots(\forall X_n)(\exists Y)(\forall y)(y \in Y \equiv \varphi(y, X_1, \ldots, X_n))$$

3) All axioms of ZF_{Fin}

(more precisely we assume that $V \models \varphi$ holds for every φ which is a formal axiom of (formalized) Zermelo-Fraenkel set theory of finite sets - cf. also another formulation in [4]).

4) Prolongation axiom

$$(\forall F)((\mathrm{Fnc}(F) \,\&\, \mathrm{Count}(F)) \rightarrow (\exists f)(\mathrm{Fnc}(f) \,\&\, F \subseteq f))$$

5) Axiom of choice

The universal class V can be wellordered

6) Axiom of cardinalities

For every two uncountable classes there is one-one mapping between them (i.e. there is only one uncountable cardinality).

In the following diagram $\longrightarrow$ denotes the existence of an interpretation of the first theory in the second one.

$$\begin{tikzcd}
AST \arrow[rr, "(1)"] & & KM_{-} + CH \arrow[dl, "(2)"] \\
& KM_{-} \arrow[ul, "(3)"] &
\end{tikzcd}$$

The arrow (2) is assured by § 2. In the following we are going to sketch proofs of the last two interpretations (detailed proofs can be found in [4]).

Interpretation (1). We can construct in $KM_- + CH$ a countable model $\mathfrak{A}$ of (formal) ZF_{Fin}. Let Z be a nontrivial ultrafilter on ω. Interpreting $*$-sets as elements of $|\mathfrak{A}^\omega/Z|$ (and $x^* \in^* y^*$ as $\mathfrak{A}^\omega/Z \models x^* \in y^*$) and $*$-classes as subsets of $|\mathfrak{A}^\omega/Z|$ (and defining $x^* \in^* X^*$ as $x^* \in X^*$ for X^* which is not a $*$-set) we get an interpretation of AST in $KM_- + CH$ (more precisely we identify x^* with $\{y^*;\ \mathfrak{A}^\omega/Z \models y^* \in x^*\}$). Since the axiom A7 in the sense of the interpretation follows from the fact that the cardinality of $|\mathfrak{A}^\omega/Z|$ is $\aleph_1$ (according to CH), the only little bit nontrivial is the axiom A5.
Let $F \subseteq |\mathfrak{A}^\omega/Z|$ & card(F) = $\aleph_0$ & $Fnc^*(F)$. Every finite subset of F (G, say) can be considered as an element of $|\mathfrak{A}^\omega/Z|$ and moreover we have $\mathfrak{A}^\omega/Z \models Fnc(G)$. Hence the existence of f with $Fnc^*(f)$ & $Set^*(f)$ & $F \subseteq^* f$ follows from the fact that $\mathfrak{A}^\omega/Z$ is $\aleph_1$-saturated.

Interpretation (3). Since AST is stronger than ZF_{Fin}, we are able to define the class N of all natural numbers. Moreover we can define FN (the class of all finite natural numbers) by

$$FN = \{n \in N;\ Fin(n)\}$$

In AST one can prove that FN with the usual operations is a model of Peano arithmetic. Moreover we obtain the formula

$$(\forall X \subseteq FN)(\exists x)(X = x \cap FN)$$

as a consequence of the axiom A5 and therefore we are able to code every subclass of FN by a set.

Hence Zbierski's construction (see [7]) gives us (using the axiom of choice) a model $\mathfrak{A}$ of ZF^- with absolute equality such that

(i) $(\forall x)(\mathrm{Count}(\{y;\ \mathfrak{A} \models y \in x\}))$

(ii) $(\forall X \subseteq |\mathfrak{A}|)(\mathrm{Count}(X) \longrightarrow (\exists x \in |\mathfrak{A}|)(X = \{y;\ \mathfrak{A} \models y \in x\}))$

and so if we interpret $*$-sets as elements of $|\mathfrak{A}|$ (and $x^* \in^* y^*$ as $\mathfrak{A} \models x^* \in y^*$) and $*$-classes as subsets of $|\mathfrak{A}|$ (and $x^* \in^* X^*$ as $x^* \in X^*$ for X^* which is not a $*$-set), we get an interpretation of KM_- in AST.

If we carefully check our construction, we see that the axiom of cardinalities was not used in it. Moreover the last construction can be made even in the theory containing the axioms 1) and 2) and weaker forms of the axioms 3) - 6) only (see [4]). In that paper it is further shown that the prolongation axiom is essential since there is no interpretation of AST in "theories without the prolongation axiom".

References

[1] R. BOYD, G. HENSEL and H. PUTNAM: A recursion-theoretic characterization of the ramified analytic hierarchy, T.A.M.S. 141(1969), 37-62

[2] P.J. COHEN: A minimal model for set theory, Bull. Am. Math. Soc. 69(1963), 537-540

[3] W. MAREK and A. MOSTOWSKI: On models of ZF set theory extendable to the models of KM theory of classes, in: Proceedings of Kiel conference, Springer LN in Math. 499(1975), 460-542

[4] A. SOCHOR: Metamathematics of alternative set theory, to appear

[5] A. SOCHOR: The Alternative Set Theory, in: Set Theory and Hierarchy Theory, A Memorial Tribute to A. Mostowski, Springer LN in Math. 537(1976), 259-272

[6] P. VOPĚNKA: Mathematics in alternative set theory, to appear

[7] P. ZBIERSKI: Models for higher order arithmetics, Bull. Acad. Pol. Sci. 19(1971), 557-562

[8] P. ZBIERSKI: A classification of interpretations , to appear

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