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ON ROUGH NORMS ON BANACH SPACES

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Abstract: Rough and strongly rough norms on Banach spaces are studied, their characterizations and duality properties are derived. Some results of M.M. Day and E.B. Leach and J.H.M. Whitfield on the existence of smooth norms are generalized. A short proof of a recent theorem of Ch. Stegall concerning Asplund spaces is given.

Key words: Asplund spaces, dentability, rough norms.

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We consider only real Banach spaces. By a subspace we mean a closed linear subspace.

Definition 1 ([10],[11]). A norm of a Banach space X is called to be rough if there is an $\varepsilon > 0$ such that for every $x \in X$ and $\delta > 0$, there are $x_1, x_2, u \in X$, $\|x_1 - x\| < \delta$, $\|x_2 - x\| < \delta$, $\|u\| = 1$ with $\|x_2\|'(u) - \|x_1\|'(u) \geq \varepsilon$, where $\|z\|'(u)$ denotes the one sided Gâteaux differential of $\|\cdot\|$ at z in the direction u, i.e.

$$\|z\|'(u) = \lim_{t \to 0+} t^{-1} (\|z + tu\| - \|z\|).$$

Remark 1. The usual norms of $C\langle 0,1\rangle$ and $\ell_1(N)$ are easily seen ([10]) to be rough (use e.g. Proposition 1).

Definition 2. If $K \subset X$ is a bounded set and $f \in X^*$, $\delta > 0$, then the set $K(f, \delta) = \{x \in K;\ f(x) \geq \sup f - \delta\}$ is called a slice of K. If $K \subset X^*$ and $f \in X$, then $K(f, \delta)$

is called a w^*-slice of K. K is dentable (w^*-dentable for $K \subset X^*$) if for every $\varepsilon > 0$ there is a slice $K(f,\delta)$ of K (w^*-slice $K(f,\delta)$ of K) with diam $K(f,\delta) < \varepsilon$. Summarizing the known facts and some considerations in this paper we easily arrive to

Proposition 1. The following properties of a given norm of X are equivalent:

(i) $\|\cdot\|$ is not rough,

(ii) B_1^* - the dual unit ball of $(X, \|\cdot\|)^*$ is w^*-dentable,

(iii) For every $\varepsilon > 0$ there is an $x \in X$, $\|x\| = 1$ with
$$\limsup_{\|y\| \to 0} \|y\|^{-1}(\|x + y\| + \|x - y\| - 2) \leq \varepsilon .$$

(iv) For every $\varepsilon > 0$ there is an $x \in X$, $\|x\| = 1$ such that whenever $f_n, g_n \in X^*$, $\|f_n\| = \|g_n\| = 1$, $\lim f_n(x) = \lim g_n(x) = 1$, then $\limsup \|f_n - g_n\| \leq \varepsilon$.

(v) Negation of: there is an $\varepsilon > 0$ such that for every $x \in X$, $\|x\| = 1$ and for every $\delta > 0$, there is a $v \in X$, $\|v\| \leq 1$ with $\|x + tv\| \geq \|x\| + \varepsilon|t| - \delta$ for any $|t| \leq 1$.

Proposition 1'. The following properties of a norm $\|\cdot\|$ on X are equivalent:

(i) $B_1 \subset X$ the unit ball of $(X, \|\cdot\|)$ is dentable,

(ii) $\|\cdot\|^*$ the dual norm of X^* is not rough,

(iii) for every $\varepsilon > 0$ there is an $x^* \in X^*$, $\|x^*\| = 1$ with $\limsup_{\|y^*\| \to 0} \|y^*\|^{-1}(\|x^* + y^*\| + \|x^* - y^*\| - 2) \leq \varepsilon$,

(iv) for every $\varepsilon > 0$, there is an $x^* \in X^*$, $\|x^*\| = 1$ such that whenever $x_n, y_n \in X$, $\|x_n\| = \|y_n\| = 1$, $\lim x_n(x^*) = \lim y_n(x^*) = 1$, then $\limsup \|x_n - y_n\| \leq \varepsilon$.

(v) Negation of: There is an $\varepsilon > 0$ such that for every

$x^* \in X^*$ $\|x^*\| = 1$ and for every $\delta > 0$ there is a $v^* \in X^*$, $\|v^*\| \leq 1$, with $\|x^* + tv^*\| \geq \|x^*\| + \varepsilon|t| - \delta$ for every $|t| \leq 1$.

Definition 3. A norm $\|\cdot\|$ of a Banach space X is said to be strongly rough if there is an $\varepsilon > 0$ such that for every $x \in X$ $\|x\| = 1$ there is a $y \in X$, $\|y\| = 1$ with $\limsup_{t \to 0+} t^1(\|x + ty\| + \|x - ty\| - 2) \geq \varepsilon$.

Remark 2. The usual norm of $\ell_1(\Gamma)$ is strongly rough if Γ uncountable (see Proposition 4).

Proposition 2. The following properties of a norm $\|\cdot\|$ of X are equivalent:

(i) $\|\cdot\|$ is not strongly rough,

(ii) for every $\varepsilon > 0$, there is an $x \in X$ $\|x\| = 1$ such that whenever $\|f_1\| = \|f_2\| = 1$, $f_1(x) = f_2(x) = 1$, then $\|f_1 - f_2\| \leq \varepsilon$.

(iii) Negation of: There is an $\varepsilon > 0$ such that for every $x \in X$ $\|x\| = 1$, there is a $u \in X$, $\|u\| = 1$ such that for every $\delta > 0$ there are $x_1, x_2 \in X$, $\|x_1\| = \|x\|_2 = 1$, $\|x_i - x\| < \delta$, $i = 1,2$ with $\|x_1\|'(u) - \|x_2\|'(u) \geq \varepsilon$.

(iv) Negation of: There is an $\varepsilon > 0$ such that for every $x \in X$ $\|x\| = 1$ there is a $v \in X$, $\|v\| \leq 1$ with $\|x + tv\| \geq$ $\geq \|x\| + \varepsilon|t|$.

Proofs follow easily by use of some ideas of E.B. Leach and J.H.M. Whitfield and standard duality arguments : First we show Proposition 1 . Propositions 2, 1′ can be proved similarly .

(iii)$\Longleftrightarrow$ (iv): is in fact contained e.g. in [2]. We sketch the proof here for the completeness:

If (iii) holds, then for every $\varepsilon > 0$ there is an $x \in X$, $\| x \| = 1$ and $\delta > 0$ with $\| x + y \| + \| x - y \| - 2 \leq \leq \varepsilon \| y \|$ for $\| y \| \leq \delta$. Thus whenever $f_n, g_n \in X^*$, $\| f_n \| = \| g_n \| = 1$, $\lim f_n(x) = \lim g_n(x) = 1$, then $f_n(x + y) + g_n(x - y) \leq \varepsilon \| y \| + 2$ for $\| y \| \leq \delta$. Therefore $| f_n(y) - g_n(y) | \leq \varepsilon \delta + 2 - f_n(x) - g_n(x) \leq 2 \varepsilon \delta$ for $n \geq n_0$ and $\| y \| \leq \delta$. Thus $\| f_n - g_n \| \leq 2 \varepsilon$ for $n \geq n_0$. If (iii) does not hold, then there is an $\varepsilon > 0$ such that for any $x \in X$, $\| x \| = 1$, there is a sequence $y_n \in X \setminus \{ 0 \}$, $\lim y_n = 0$, with $\| x + y_n \| + \| x - y_n \| - 2 \geq \varepsilon \| y_n \|$. Then taking $f_n, g_n \in X^*$, $\| f_n \| = \| g_n \| = 1$ such that
$f_n(x + y_n) = \| x + y_n \|$, $\quad g_n(x - y_n) = \| x - y_n \|$
we have $\lim f_n(x) = \lim g_n(x) = 1$, $f_n(x + y_n) + g_n(x - y_n) \geq \geq 2 + \varepsilon \| y_n \|$ and hence $(f_n - g_n)(y_n) \geq 2 - f_n(x) - g_n(x) + + \varepsilon \| y_n \| \geq \varepsilon \| y_n \|$, so, (iv) does not hold.

(ii) $\Longleftrightarrow$ (iv) is easily seen;

(i) $\Longrightarrow$ (iv): implicitly contained in [11], p. 122: If (iv) does not hold, then there is an $\varepsilon > 0$ such that for any $x \in X$, $\| x \| = 1$, there is a sequence $f_n, g_n \in X^*$, $\| f_n \| = = \| g_n \| = 1$, $\lim f_n(x) = \lim g_n(x) = 1$ and a sequence $u_n \in X$, $\| u_n \| = 1$, with $(f_n - g_n)(u_n) \geq \varepsilon$. So, if $0 < \eta < \varepsilon/2$ is an arbitrary number, $x \in X$, $\| x \| = 1$ and $t > 0$, then for $n \geq \geq n_0$, $f_n(x) > 1 - \eta^2/4$, $g_n(x) > 1 - \eta^2/4$, and thus $\| x + + tu_n \| \geq f_n(x + tu_n) \geq 1 - \eta^2/4 + t f_n(u_n)$. From this and elementary convexity properties it follows that
$t \| x + tu_n \|' (u_n) \geq \| x + tu_n \| - \| x \| \geq t f_n(u_n) - \eta^2/4$, so
$\| x + tu_n \|' (u_n) \geq f_n (u_n) - \eta^2/4 t$.
Similarly, $- \| x - t u_n \|' (u_n) \geq - g_n(u_n) - \eta^2/4 t$.
Thus, choosing $t \in (0, \eta)$ such that $\eta^2/2 t < \eta$, we finally

have $\|x + t\, u_n\|'(u_n) - \|x - t\, u_n\|'(u_n) \geq (f_n - g_n)(u_n) - $
$- \eta^2/2t > \varepsilon - \eta > \varepsilon/2$, so (i) does not hold.

(v) $\Longrightarrow$ (i) is proved in [11]: If $\|\cdot\|$ is rough with some $\varepsilon > 0$, then given $x \in X$, $\|x\| = 1$, $\eta > 0$, we can choose $x_1, x_2 \in X$, $\|x_i - x\| < \eta$, and $u \in X$, $\|u\| = 1$, with $\|x_2\|'(u) - \|x_1\|'(u) \geq \varepsilon$. Then putting $v = u -$
$- [(\|x_1\|'(u) + \|x_2\|'(u))/2]\, x$, we have $\|v\| \leq 2$ and if $s = 1 - t\,(\|x_1\|'(u) + \|x_2\|'(u))/2$, then $0 < s < 2$ for $t \in$
$\in (0,1)$ and

$$\|x + tv\| = \|sx + tv\| = \|s(x_2 + (t/s)u) + s(x - x_2)\| \geq$$
$$\geq s(\|x_2\| + \tfrac{t}{s}\|x_2\|'(u)) - s \cdot \eta/4 \geq s\|x\| + t\|x_2\|'(u) -$$
$$- s\eta/2 = \|x\| + (t/2)(\|x_2\|'(u) - \|x_1\|'(u)) - s\eta/2 \geq$$
$$\geq \|x\| + (t \cdot \varepsilon/2) - \eta .$$

If $-1 < t < 0$, we use x_1 instead of x_2 and have

$$\|x + tv\| \geq \|x\| + (-t/2)(\|x_2\|'(u) - \|x_1\|'(u)) - s\eta/2 \geq$$
$$\geq \|x\| + (|t|/2)\varepsilon - \eta .$$

So, (v) $\Longrightarrow$ (i).

If (v) does not hold, then there is an $\varepsilon > 0$ such that for any $x \in X$, $\|x\| = 1$ and for any $n \in N$ (N positive integers), there is a $v_n \in X$, $\|v_n\| \leq 1$ with $\|x + tv_n\| \geq \|x\| + \varepsilon|t| -$
$- n^{-2}$ for every $|t| \leq 1$. So, putting $t = n^{-1}$, we have $n(\|x + n^{-1} v_n\| + \|x - n^{-1} v_n\| - 2) \geq \varepsilon - n^{-1} > \varepsilon/2$ for large n, so, (iii) does not hold (see [11]).

For the proof of the other statements which can be proved similarly, we mention only a few remarks.

1. In Prop. 1', non (iii) $\Longrightarrow$ non (iv): choose f_n, g_n as in Prop. 1 only $f_n(x + y_n) \geq \|x + y_n\| - n^{-1}\|y_n\|$, similarly g_n.

2. In Prop. 2 non (i) $\Longrightarrow$ non(ii): use limit points of f_n, g_n constructed in the corresponding part of Prop 1.

3. In Prop. 2 (ii) $\Longrightarrow$ (iii) follow (i) $\Longrightarrow$ (iv) in Prop. 1 choosing $f, g \in B_1^*$, $f(x) = g(x) = 1$, $u \in X$, $\|u\| = 1$ with $(f - g)(u) \geq \varepsilon$.

4. In Prop. 2 (iv) $\Longrightarrow$ (i): If $\|\cdot\|$ is strongly rough for some $\varepsilon > 0$, then given $x \in X$, $\|x\| = 1$, there is a $u \in X, \|u\| = 1$ such that for any $n \in N$, there are $x_1^n, x_2^n \in X$, $\|x_i^n\| = 1$, $\|x_i^n - x\| \leq n^{-1}$ with $\|x_1^n\|'(u) - \|x_2^n\|'(u) \geq \varepsilon$. Then as shown in [11] (see Prop. 1) putting $v_n = u - [(\|x_1^n\|'(u) + \|x_2^n\|'(u))/2] \cdot x$, we have $\|v_n\| \leq 2$ and $\|x + tv_n\| \geq \|x\| + (|t|/2)\varepsilon - 4/n$, for every $|t| \leq 1$. Since $v_n \in sp(x,u)$ (the linear hull of $\{x,u\}$), $\|v_n\| \leq 2$ choosing a limiting point v of $\{v_n\}$ in the norm topology, we have $\|v\| \leq 2$ and $\|x + tv\| \geq \|x\| + (|t|/2)\varepsilon$ for every $|t| \leq 1$.

Remark 3. The condition (ii) in Proposition 1 cannot generally be replaced by that of dentability of $B_1^* \subset X^*$, since as mentioned above, the usual norm of $C\langle 0,1\rangle$ is rough and $B_1^* \subset C^*\langle 0,1\rangle$ is easily seen (and certainly known) to be dentable: Considering $C^*\langle 0,1\rangle \cong V\langle 0,1\rangle$ the space of all functions on $\langle 0,1\rangle$ with bounded variation, with $f(0) = 0$ and f right continuous on $(0,1)$, we easily see that for e.g. $\delta_0 = \chi_{(0,1\rangle}$ the characteristic function of $(0,1\rangle$, $\delta_0 \notin \overline{\text{conv}}(B_1^* \setminus B_\varepsilon(\delta_0))$ (for every $\varepsilon > 0$): If for some $\varepsilon > 0$, $\delta_0 \in \overline{\text{conv}}(B_1^* \setminus B_\varepsilon(\delta_0))$, then there are $v_1, \dots, v_n \in B_1^* \setminus B_\varepsilon(\delta_0)$ and $c_1, \dots, c_n \geq 0$, $\sum c_i = 1$, such that $\text{var}(\delta_0 - \sum c_i v_i) \leq \varepsilon/8$. Then for every $t \in (0,1\rangle$ there is an $i \in \{1, \dots, n\}$ with $v_i(t) \geq 1 - \varepsilon/8$.

Thus there is a $j \in \{1,\dots,n\}$ with $\limsup v_j(t) \geq 1 - \varepsilon/8$.

Therefore for every $a > 0$, $\operatorname*{var}_{\langle 0,a\rangle} v_j \geq 1 - \varepsilon/8$ and thus, $\operatorname*{var}_{\langle a,1\rangle} v_j \leq \varepsilon/8$.

Therefore, $\operatorname*{var}_{\langle a,1\rangle} (v_j - \sigma_0) \leq \varepsilon/8$ for each $a \in (0,1\rangle$. Furthermore, for every $t \in (0,1\rangle$, $|v_j(t) - 1| < \varepsilon/2$, otherwise $\operatorname*{var}_{\langle \gamma,1\rangle} v_j > \varepsilon/8$ for some $\gamma \in (0,1\rangle$. So, $\sup_{\langle 0,1\rangle} |v_j - \sigma_0| \leq \varepsilon/2$ and for every $a \in (0,1\rangle$, $\operatorname*{var}_{\langle a,1\rangle} (v_j - \sigma_0) \leq \varepsilon/8$. Therefore, $\operatorname*{var}_{\langle 0,1\rangle} (v_j - \sigma_0) < \varepsilon$, a contradiction.

Since the unit ball of $C\langle 0,1\rangle$ is not even smoothable (for the definition and this result see [9]), Remark 3 answers negatively a part of the Question 4 in [9] the other part was recently answered in [12].

Remark 4. There is an equivalent non rough norm on $\ell_1(\Gamma)$, Γ uncountable, with no point of Gâteaux differentiability. This is easy to see by a slight modification of a construction of M. Edelstein ([7], p. 111):

Let $\{e_\gamma, f_\gamma\}$ be the usual unit vector basis in $\ell_1(\Gamma)$ (f_γ the biorthogonal functionals), $\{f_n\}$, $n \in N$ a sequence of disjoint elements of $\{f_\gamma\}$.

Let $D_n = x \in \ell_1^*(\Gamma) \simeq m(\Gamma);\ \|x - (2 - 2^{1-n})f_n\| \leq 2^{1-n}\}$.

Let $C_0 = \bigcup_{n=1}^{\infty} (D_n \cup (-D_n))$ $C_1 = \operatorname{conv} C_0$, $C = w^*$ closure of C_1. Then

(i) C is a w^* compact absolutely convex body in $\ell_1^*(\Gamma)$, $C \subset 2B_1^*$;

(ii) C is w^* dentable;

(iii) C has no w^* exposed point ($x \in K \subset X^*$ is w^* exposed point of K if there is an $f \in X$ with $f(x) > f(y)$ for every $y \in K$ $y \neq x$).

To see (ii) it clearly suffices to show that C_1 is

w^* dentable. For this purpose, following [7], p. 113, denote for $n \in N$: $H_n = \{x \in C_1;\ x(e_n) \geq 2 - 2^{1-n}\}$. Then H_n is a w^* slice of C_1. Fix now $n \in N$ and compute the diameter of H_n. Consider an arbitrary $h \in H_n$, $h = \sum_{i=1}^{k} c_i h_i$, $h_i \in \bigcup_{j=1}^{\infty} D_j \cup$ $\cup (-D_j)$, $c_i \geq 0$, $\sum c_i = 1$. Then write $h = \sum_{h_i \in D_m} c_i h_i +$ $+ \sum_{h_i \notin D_m} c_i h_i$. Easily, if $h_i \notin D_n$, then $h_i(e_n) \leq 1$. So, $2 - 2^{1-n} \leq$ $\leq (\sum_{h_i \in D_m} c_i h_i + \sum_{h_i \notin D_m} c_i h_i)(e_n) \leq \sum_{h_i \in D_m} c_i h_i(e_n) + \sum_{h_i \notin D_m} c_i =$ $= \sum_{h_i \in D_m} c_i (h_i(e_n) - 1) + 1 \leq \sum_{h_i \in D_m} c_i + 1$. Thus $\sum_{h_i \in D_m} c_i \geq 1 -$ $- 2^{1-n}$. Therefore, if $h \in H_n$, then for $n \geq 2$,

$$\| h - (2 - 2^{1-n}) f_n \| \leq \| (\sum_{h_i \in D_m} c_i)^{-1} \sum_{h_i \in D_m} c_i h_i - (2 - 2^{1-n}) f_n \| + [(\sum_{h_i \in D_m} c_i)^{-1} - 1] \cdot \| \sum_{h_i \in D_m} c_i h_i \| + \sum_{h_i \notin D_m} c_i h_i \leq 2^{1-n} + 2 [(1 - 2^{1-n})^{-1} - 1] + 2 \cdot 2^{1-n}.$$

To see (iii) let us first observe that $C_1 = \bigcup_j (D_j \cup$ $\cup (-D_j))$ is w^* compact. Thus any w^* exposed point of C lies in C_1 (Milman). But D_j have no w^* exposed points, since the usual norm of $\ell_1(\Gamma)$ is nowhere Gâteaux differentiable, if Γ is uncountable (cf. e.g. Proposition 4).

Before proceeding let us recall that a function f is said to be Gâteaux differentiable at $x \in X$ if $\lim t^{-1} [f(x + th) - f(x)]$ exists for each $h \in H$ and is a continuous linear functional on X.

We will need the following version of Lemma 3.1 in [11].

<u>Lemma 1</u>. Let a Banach space X admit a strongly rough equivalent norm $\| \cdot \|$. Then if f is a continuous Gâteaux differentiable real-valued function on X, $f(0) = 0$, then there is an $x \in X$, $1 \leq \|x\| < 2$ with $f(x) \leq \|x\|$.

Proof. The same as that of Lemma 3.1 of [11]; we sketch it here only for the completeness:

Choose a sequence $x_n \in X$, $x_o = 0$ such that

(i) $f(x_n) \leq \| x_n \|$;

(ii) $\| x_{n+1} - x_n \| \leq 1$;

(iii) $\| x_{n+1} \| \geq \| x_n \| + (\varepsilon/8) \| x_{n+1} - x_n \|$;

(iv) $\| x_{n+1} - x_n \| \geq \frac{1}{2} M_n = \frac{1}{2} \sup \| y - x_n \|$, $y \in X$, $y = x_{n+1}$

satisfies (i),(ii),(iii), where ε is from strong roughness of $\| \cdot \|$.

First observe that it suffices to show that for some n, $\| x_n \| \geq 1$. Supposing $\| x_n \| \leq 1$ for each n, we have $\{ \| x_n \| \}$ convergent and thus $\{ x_n \}$ convergent ((iii)) to some $z \in X$, $\| z \| \leq 1$, $f(z) \leq \| z \|$. By strong roughness of $\| \cdot \|$, there is for z a $v \in X$, $\| v \| \leq 1$ with $\| z + tv \| \geq \| z \| + |t| \varepsilon$ for $|t| \leq \| z \|$. By Gâteaux differentiability of f at z, there is $\delta \in (0, \| z \|)$ with $f(z + tv) \leq f(z) + f'(z)(tv) + (\varepsilon/8)|t|$ for each $|t| < \delta$. Choosing $t = \pm \delta/2$ depending on the sign $f'(z)(v)$ we have $f(z + (\delta/2)v) \leq f(z) + \varepsilon\delta/16$ and $\| z + (\delta/2) v \| \geq \| z \| + \varepsilon\delta/2$. From this it follows that for large n, $M_n \geq \varepsilon\delta/2$, so, $\| x_{n+1} - x_n \| \geq \varepsilon\delta/4$, a contradiction.

Corollary. If X admits a continuous, Gâteaux differentiable real valued function f with bounded nonempty support, then X does not admit any strongly rough equivalent norm.

Proof. ([11]) If X admits such an f, then we can easily produce a continuous Gâteaux differentiable function g with $g(0) = 0$ and $g(x) = 2$ for $\| x \| \geq 1$ and receive a contradiction with Lemma 1.

Applications

Definition 4. A Banach space X is called an Asplund space if each continuous convex function f on X is Fréchet differentiable on a dense G_δ subset of X.

Remark 5. Reflexive spaces, spaces with separable dual (or more generally spaces with WCG dual) are Asplund spaces ([13],[3],[8]).

We can now easily show

Proposition 3. A Banach space X is an Asplund space iff X does not admit any equivalent rough norm.

Proof. Easily, rough norm is nowhere Fréchet differentiable (use e.g. Proposition 1).
If X is not an Asplund space, then X contains a w^*-compact convex set (we may assume K to be absolute convex body), which is not w^* dentable ([13]). This produces by duality (Proposition 1) a rough norm on X.

From the preceding Proposition we can easily deduce the following result of Ch. Stegall:

Theorem (Stegall). A Banach space X is an Asplund space iff each separable subspace $Y \subset X$ has a separable dual.

Proof. If X is not an Asplund space, then X admits an equivalent rough norm $\|\cdot\|$ (Proposition 3). Then, easily, see [11], p. 125 , there is a separable subspace $Y \subset X$ on which $\|\cdot\|$ is rough: Construct separable subspaces C_n, $C_n \subset C_{n+1}$ and countable dense sets $D_n \subset C_n$ with the property that C_{n+1} contains all the corresponding v 's from Proposition 1 (v) for each $x \in D_n$ and δ's positive rationals.

Then easily, $\overline{\bigcup_n C_n}$ is the desired subspace Y. Y admits no equivalent Fréchet smooth norm, by the result of Leach and Whitfield mentioned above (before Lemma 1). So, Y^* is nonseparable (Kadec, Klee, Restrepo, cf. e.g. [5]).

If there is a separable subspace $Y \subset X$ with Y^* nonseparable, then Y admits an equivalent rough norm (Leach, Whitfield [11]). This can be seen as follows: Clearly, there is a $\delta > 0$ and an uncountable subset $D \subset B_1^*$ such that $f, g \in D$ $f \neq g$ imply $\|f - g\| > \delta$. Since B_1^* satisfies the second axiom of countability relative to the w^* topology, by deleting at most countably many points of D we receive $D_1 \subset D$ all points of which are w^* condensation points of D_1. Thus D_1 is not w^* dentable (here we followed [13]). Therefore there is a w^* compact absolutely convex body $K \subset Y$ which is not w^* dentable ($K = B_1^* + \overline{\mathrm{conv}}^{w^*}(D_1 \cup (-D_1))$). By duality (Proposition 1) we receive an equivalent rough norm on Y. So, Y is not an Asplund space and neither is therefore X ([13]).

In [4], M.M. Day proved that $\ell_1(\Gamma)$, Γ uncountable and $m(N)$ admit no equivalent Gâteaux differentiable norms. We state the following generalization of his results:

Proposition 4. $\ell_1(\Gamma)$, Γ uncountable and $m(N)$ admit no continuous, Gâteaux differentiable function with bounded nonempty support.

Proof. By Lemma 1 it suffices to show that both spaces admit strongly rough norms. For $\ell_1(\Gamma)$ such a norm is the usual one: Given $x \in \ell_1(\Gamma)$, $\|x\| = 1$ choose $i_0 \in \Gamma \setminus \setminus \{\mathrm{supp}\, x\}$. Let

$f_i = \mathrm{sgn}\, x_i$ for $i \in \mathrm{supp}\, x$, $f_{i_0} = 1$, $f = 0$ elsewhere

$g_i = \operatorname{sgn} x_i$ for $i \in \operatorname{supp} x$, $g_{i_0} = -1$, $g = 0$ elsewhere.

Then $\|f\| = \|g\| = 1$ $f(x) = g(x) = 1$, $\|f - g\| = 1$.

For m (N) consider the following norm ([15])

$|||x||| = |x| + \|x\|$, where $|x|$ means the usual norm of m (N) and $\|x\| = \limsup |x_n|$. We prove that $|||x|||$ is strongly rough.

If $x \in X$, $\|x\| = 0$, then for $h_i = (-1)^i$ and $t > 0$,

$|||x \pm th||| = |x \pm th| + t$. Thus

$$\limsup_{t \to 0+} t^{-1} [|||x + th||| + |||x - th||| - 2|||x|||] =$$

$$= \limsup_{t \to 0+} t^{-1} [|x + th| + |x - th| - 2|x| + 2t] \geq 2.$$

If $\|x\| > 0$ and $\lim |x_{n_k}| = \|x\|$, x_{n_k} have the same sign (say 1), let $h_{n_k} = (-1)^k$, $h_i = 0$ for $i \neq n_k$, $k = 1,2,\ldots$.

Then $\limsup_{t \to 0+} t^{-1}[|x + th| + |x - th| - 2|x| + \|x + th\| +$
$+ \|x - th\| - 2\|x\|] \geq \limsup_{t \to 0+} t^{-1}[\|x + th\| + \|x - th\| -$
$- 2\|x\|] = \limsup_{t \to 0+} t^{-1}[\|x\| + t + \|x\| + t - 2\|x\|] \geq 2.$

We finish the paper with a simple example and one question.

First we will need the following

<u>Definition 5</u>. A subspace $Y \subset X^*$ is called 1-norming if for each $x \in X$, $\|x\| = \sup (f(x), f \in Y, \|f\| \leq 1)$.

Example. There is a 1-norming subspace $Y \subset m(N)$ on which the usual norm of m(N) is rough.

(Since the norm of m(N) is Fréchet differentiable on a dense G_δ subset of m(N) it is not rough on the whole space.)

Let Y_1 be the family of all periodic functions on N. Then, easily, $Y = \operatorname{cl} Y_1$ is a closed linear 1-norming subspace of

$m(N)$.

If $y \in Y$ $\|y\| = 1$ and $\varepsilon > 0$, then finding $y^1 \in Y_1$ with $\|y^1 - y\| < \varepsilon$ we see that there are two indexes $i,j,i \neq j$ with $\pm y_i > 1 - \varepsilon$, $\pm y_j > 1 - \varepsilon$. From this we derive that the norm of $m(N)$ is rough on Y and that the unit ball of $\ell_1(N)$ is not Y dentable (i.e. by slices given by functionals from Y) This can be compared with the result of Charles Stegall who proved that if any bounded subset of X^* is dentable, then any bounded subset of X^* is w^* dentable ([16]).

The following seems to be an open problem

Question. Suppose that any w^* compact convex subset of X^* is dentable. Is then X necessarily an Asplund space?

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