Commentationes Mathematicae Universitatis Carolinae

Josef Mlček

End-extensions of countable structures and the induction schema

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 2, 291--308

Persistent URL: http://dml.cz/dmlcz/105853

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,2 (1978)

END-EXTENSIONS OF COUNTABLE STRUCTURES AND THE INDUCTION SCHEMA

J. MIČEK, Praha

Abstract: We present necessary and sufficient conditions for the existence of some types of end-extensions of countable models. We discuss relations between these conditions and the induction schema.

Key words: Countable model, end-extension, induction schema.

AMS: 02H05, 02H15

Introduction. In this paper we shall present some results concerning end-extensions of countable structures and relations between the end-extensions and the induction axioms. In § 1 we present a general form of the theorem on end-extensions of countable model $\mathfrak{A} \models U$, where U is a theory of the directed antisymmetric binary relation with an arbitrary large transitive element.

In § 2, we present a necessary and sufficient condition for the existence of a Π_n end-extension of a countable model $\mathfrak{A} \models U$. The condition in question concerns the validity of the schema $H(\Pi_n)$ in $\mathfrak{A}$. (For $H(\varphi)$ see 2.0.)

In § 3, we discuss relations between the induction schema for Σ_n and the schema $H(\Pi_n)$. Some consequences are introduced in § 4. As a consequence of the results presented we obtain the McDowell-Specker's theorem and the well known statement that each countable model of ZF has an elementary end-extension.

The present work has been stimulated by the Vopěnka's question as to whether every $\aleph_1$-like model of the arithmetic with the induction schema for Δ_0 formulas only is a model of Peano arithmetic.

§ 0. Notation and definitions.

0.0. By a language we mean a first-order predicate language with = . The symbol a designates a sequence $a_1, \ldots$ $\ldots, a_n$. If $a_i \in A$ for $i = 1, \ldots, n$, we write $\bar{a} \in \bar{A}$. For a language L and a set A such that $L \cap A = 0$ we denote L(A) the language obtained from L by adding all the $a \in A$ as constants. Fm(L) (AFm(L) res.) is the set of all formulas (atomic formulas resp.) of L. By $\mathfrak{A} \models L$ we mean that $\mathfrak{A}$ is a structure for L. The name of $a \in A$ is $\underline{a}$. We suppose that $L \cap \underline{A} = 0$ (where $\underline{A} = \{\underline{a};\ a \in A\}$) and we have $\underline{\mathfrak{A}} \models L(\underline{A})$. For $\Gamma \subseteq$ $\subseteq Fm(L)$ we put $\Gamma(A) = \{\varphi(\bar{a});\ \varphi(\bar{x}) \in \Gamma$ and $\bar{x} \subseteq Fv(\varphi)\}$, where $Fv(\varphi) = \{x;\ x$ is a free variable of $\varphi\}$. We have $\Gamma(A) \subseteq Fm(L(A))$. For $\mathfrak{A} \models L$, $\mathfrak{B} \models L$ and $\Gamma \subseteq Fm(L)$ we write $\mathfrak{A} \subset_\Gamma \mathfrak{B}$ if $\mathfrak{A}$ is a Γ-substructure of $\mathfrak{B}$. Instead of $\mathfrak{A} \subset_{Fm} \mathfrak{B}$ ($\mathfrak{A} \subset_{AFm} \mathfrak{B}$ resp.) we write $\mathfrak{A} \prec \mathfrak{B}$ ($\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$) ($\mathfrak{A} \subset \mathfrak{B}$ resp. ($\mathfrak{B}$ is an extension of $\mathfrak{A}$)).

For $F: Fm(L) \longrightarrow Fm(L)$, $\Gamma \subseteq Fm(L)$ we define $F(\Gamma) =$ $= \{F(\varphi);\ \varphi \in \Gamma\}$. By $\forall\Gamma$ we mean the set $\{(\forall \bar{x})\varphi;$ $\varphi \in \Gamma$ and no member of $\bar{x}$ is bound in $\varphi\}$. Analogously we define $\exists\Gamma$, $\forall\exists\Gamma$, etc. If $T\subseteq Fm(L)$, $\Gamma\subseteq Fm(L)$ we put $\Gamma^T = \{\varphi$; there is some $\psi\in\Gamma$ such that $T\vdash \varphi\equiv\psi\}$. We write sometimes Γ instead $\Gamma^{log.ax.}$. For $T,S\subseteq Fm(L)$ we write $T\prec S$ to mean that $T\vdash\varphi$ implies $S\vdash\varphi$. $T\equiv S$ denotes $T\prec S$ and $S\prec T$.

The variables for natural numbers are i,j,k,m,n.

0.1. Let $<$ be a binary predicate, $<\in L$. We denote by $\Delta_0(L) = \Delta_0$ the set of limited formulas of L (w.r.t. $<$). We put $\Pi_0 = \Sigma_0 = \Delta_0$, $\Pi_{n+1} = \forall\Sigma_n$, $\Sigma_{n+1} = \exists\Pi_n$. Instead of $\mathfrak{A}\subset_{\Pi_n}\mathfrak{B}$ we write $\mathfrak{A}\subset_n\mathfrak{B}$.

We write $(\check{\exists} x)\varphi(x)$ instead of $(\forall y)(\exists x)(y< x \,\&\, \varphi(x))$ where x,y do not occur bound in φ .

$\Gamma\subseteq Fm(L)$ is <u>closed under limited quantification</u> ($Clq(\Gamma)$) if $\varphi\in\Gamma$ implies $(\exists x<y)\varphi\in\Gamma$, $(\forall x<y)\varphi\in\Gamma$. Evidently $Clq(\Gamma)$ implies $Clq(\Gamma\cup\neg(\Gamma))$.

If $\mathfrak{A}$, $\mathfrak{B}\models L$, $\mathfrak{A}\subset\mathfrak{B}$ and $d\in B$ we write $A<d$ to mean that $a\in A$ implies $a\overset{\mathfrak{B}}{<} d$.

0.2. <u>Definition</u>. Let $\mathfrak{A}$, $\mathfrak{B}$ be structures for L ($<\in L$), $\Gamma\subseteq Fm(L)$. $\mathfrak{B}$ is a (proper) Γ end-extension of $\mathfrak{A}$, $\mathfrak{A}\Subset_\Gamma\mathfrak{B}$, if

(1) $\mathfrak{A}\subset_\Gamma\mathfrak{B}$ and $A\neq B$

(2) if $a\in A$, $b\in B$ and $b\overset{\mathfrak{B}}{<} a$ then $b\in A$

(3) there is a $d\in B - A$ such that $A<d$.

We write $\mathfrak{A}\Subset_n\mathfrak{B}$ for $\mathfrak{A}\Subset_{\Pi_n}\mathfrak{B}$. Instead of Fm end-extension (end-extension resp.) we say elementary end-extension (end-extension resp.). We write $EE_\Gamma(\mathfrak{A})$ ($EE_n(\mathfrak{A})$, $EE_\omega(\mathfrak{A})$ resp.) to mean that there is a $\mathfrak{B}$ such that $\mathfrak{A}\Subset_\Gamma\mathfrak{B}$ ($\mathfrak{A}\Subset_n\mathfrak{B}$, $\mathfrak{A}\prec\!\!\cdot\,\mathfrak{B}$ resp.).

§ 1. End-extension theorem of countable structures

The main theorem of this section is formulated in 1.4.

1.0. Let L be a language with a binary predicate $<$, We denote by $Tr(x)$ the formula $(\forall y<x)(\forall z<y)(z<x)$. (The x is transitive.)

U is the theory in L with axioms:

$$(\forall x,y)(\exists z)(x<z \,\&\, y<z)$$
$$(\forall x)(\exists y)(x<y \,\&\, Tr(y))$$
$$(\forall x,y)(x<y \rightarrow \neg(y<x)).$$

We have $U \vdash x<y \rightarrow x \neq y$ and, for every $\varphi \in Fm(L)$,

$$U \vdash (\forall \bar{x})\varphi \equiv (\forall x)(\forall \bar{x}<x)\varphi$$
$$U \vdash (\exists \bar{x})\varphi \equiv (\exists x)(\exists \bar{x}<x)\varphi$$

1.1. For $\varphi(x,y) \in Fm(L)$ we denote by $E(\varphi(x,y))$ the general closure of the formula

$$(\forall u)((\check{\exists} x)(\exists y<u)(Tr(x) \,\&\, \varphi(x,y)) \rightarrow$$
$$\rightarrow (\exists y<u)(\check{\exists} x)(Tr(x) \,\&\, \varphi(x,y)))$$

where u does not occur in φ. $E(\varphi)$ designates $E(\varphi(x,y))$ for some x,y.

1.2. In the sequel, Φ will be a set of L-formulas closed under $\&$, $\neg$, subformulas, and containing all atomic formulas.

1.3. Lemma. Let $\mathfrak{A} \models U$ and, $D \notin L(A)$. Then there is a consistent Theory $\tilde{U}$ in the language $L(\underline{A}) \cup \{D\}$ such that

(0) $\tilde{U}$ is a set of $\forall\exists\Phi(\underline{A} \cup \{D\})$ sentences and $Tr(D) \in \tilde{U}$

(1) if $\mathfrak{B} \models \tilde{U}$, then $\mathfrak{A} \subset_{\forall\exists\Phi} \mathfrak{B}$ (up to isomorphism)

(2) if $\mathfrak{B} \models \tilde{U}$, then $D^{\mathfrak{B}} \in B - A$ and $a \overset{\mathfrak{B}}{<} D^{\mathfrak{B}}$ for every $a \in A$ (i.e. $A < D^{\mathfrak{B}}$),

(3) if $\varphi(x) \in \exists \Phi(\underline{A})$ with exactly one free variable x then $\varphi(D)$ is consistent with $\tilde{U}$ iff $\mathfrak{A} \models (\check{\exists} x)(Tr(x) \,\&\, \& \, \varphi(x))$.

Proof. We put $T = \{\psi ; \psi$ is a $\forall \exists \Phi(\underline{A})$ sentence and $\mathfrak{A} \models \psi\}$ and $\tilde{U} = T \cup \{\varphi(D); \varphi \in \exists \Phi(\underline{A})$ with exactly one free variable x and

$$\mathfrak{A} \models (\exists z)(\forall x > z)(Tr(x) \rightarrow \varphi(x))\} \cup \{Tr(D)\} .$$

If $\varphi_1(D), \ldots, \varphi_n(D) \in \tilde{U}$ we have $\mathfrak{A} \models (\exists z)(\forall x > z)(Tr(x) \rightarrow \rightarrow \varphi_1(x) \,\&\, \ldots \,\&\, \varphi_n(x))$. Taking a conveniently large $a \in A$ as the interpretation of D we have $\mathfrak{A} \models T \cup \{\varphi_1(D) \,\&\, \ldots \ldots \,\&\, \varphi_n(D) \,\&\, Tr(D)\}$. Thus U is consistent. (0) and (1) are evident. We prove (2): If $\mathfrak{B} \models \tilde{U}$ and $a \in A$ we have $\mathfrak{A} \models (\exists z)(\forall x > z)(Tr(x) \rightarrow \underline{a} < x)$. Thus $\{\underline{a} < D; a \in A\} \subseteq \tilde{U}$. From $D^{\mathfrak{B}} \in A$ it follows that $a < D^{\mathfrak{B}}$ and $D^{\mathfrak{B}} < a$ for some $a \in A$, which is a contradiction. We shall prove (3): From $\mathfrak{A} \models (\check{\exists} x)(Tr(x) \,\&\, \varphi(x))$, the compactness theorem and the properties of $<$ it follows that $\tilde{U} \cup \{\varphi(D)\}$ is consistent. Conversely, let $\varphi(D)$ be consistent with $\tilde{U}$ and let $\varphi \in \in \exists \Phi(\underline{A})$ have exactly one free variable x. If $\mathfrak{A} \models \neg (\check{\exists} x)(Tr(x) \,\&\, \varphi(x))$ then $\mathfrak{A} \models (\exists z)(\forall x > z)(Tr(x) \rightarrow \rightarrow \neg \varphi(x))$. Let $a \in A$ be such that $\mathfrak{A} \models (\forall x > \underline{a})(Tr(x) \rightarrow \rightarrow \neg \varphi(x))$. We have $(\forall x > \underline{a})(Tr(x) \rightarrow \neg \varphi(x)) \in \forall \Phi(\underline{A})$. Let $\mathfrak{M}$ be a model of $\tilde{U} \cup \{\varphi(D)\}$. From $\mathfrak{A} \subset_{\forall \exists \Phi} \mathfrak{M}$ we obtain $\mathfrak{M} \models (\forall x > \underline{a})(Tr(x) \rightarrow \neg \varphi(x))$. But $\mathfrak{M} \models \underline{a} < D \,\&\, Tr(D)$. Consequently $\mathfrak{M} \models \neg \varphi(D)$, which is a contradiction.

1.4. Theorem. Let L be a countable language with a binary predicate $<$. Let $\Phi \subseteq Fm(L)$ be as in 1.2 and let $\mathfrak{A}$ be a countable model of U.

(i) If $\mathfrak{A} \models E(\exists \Phi)$ then $EE_{\forall\exists\Phi}(\mathfrak{A})$.

(ii) If $Tr \in \Phi$ and $EE_{\forall\exists\Phi}(\mathfrak{A})$ then $\mathfrak{A} \models E(\exists \Phi)$.

(iii) If $\mathfrak{A} \models (\forall x)Tr(x)$ then $EE_{\forall\exists\Phi}(\mathfrak{A})$ iff $\mathfrak{A} \models E(\exists \Phi)$.

(iv) If $\Delta_0 \subseteq \Phi$ then $EE_{\forall\exists\Phi}(\mathfrak{A})$ iff $\mathfrak{A} \models E(\exists \Phi)$.

Proof. (iv) follows immediately from (i), (ii). We prove (i). First, for a set Γ of formulas we denote by $\forall\bigvee\exists\Gamma$ the set of formulas

$$(\forall \bar{x}) \bigvee_{n \in \omega} (\exists \bar{y}_n)(\psi_{n1} \& \ldots \psi_{nj_n}); \quad \psi_{ij} \in \Gamma .$$

We shall use the following formulations of the omitting types theorem (c.f. [K]): Let J be a countable language, let $\Gamma \subseteq Fm(J)$ contain all atomic formulas and be closed under subformulas. Let C be a set of new constants. Let $T \subseteq \subseteq \forall\exists\Gamma$ be a consistent set of sentences and let

$$\varphi_n = (\forall x_1 \ldots x_{m_n}) \psi_n(x_1 \ldots x_{m_n})$$

be a countable sequence of $\forall\bigvee\exists\Gamma$ sentences. Suppose that for each n, each finite set $p \subseteq \Gamma(C)$ sentences, which is satisfiable in some model of T and each m_n-tuple $\bar{c} \in C^{m_n}$, the set $p \cup \{\psi_n(\bar{c})\}$ is satisfiable in some model of T. Then there is a countable model of $T \cup \{\varphi_n\}_n$.

Let $\tilde{U}$ be as in 1.3. We prove that there is a countable model of the theory $\tilde{U} \cup \{(\forall x)(x < \underline{a} \longrightarrow \bigvee_{b \in A} x = \underline{b};\ a \in A\}$. We have $(\forall x)(x < \underline{a} \longrightarrow \bigvee_{b \in A} x = \underline{b}) \equiv (\forall x)(\neg(x < \underline{a}) \vee \vee \bigvee_{b \in A} x = \underline{b})$. Thus it is sufficient to show that:

(*) if $\varphi(D,\bar{c},e)$ is a sentence from $\Phi(\underline{A} \cup \{D\} \cup C)$, $\bar{c} \in \bar{C}$, $e \in C$ and $\varphi(D,\bar{c},e)$ is consistent with $\tilde{U}$ then for every $a \in A$, there exists a model of $\tilde{U}$ in which $\varphi(D,\bar{c},e) \& (e < \underline{a} \longrightarrow \bigvee_{b \in A} e\ \ \underline{b})$ holds.

Let $\varphi(D,\bar{c},e) \& e < \underline{a}$ hold in a model of $\tilde{U}$. The formula $(\exists y<\underline{a})(\exists \bar{z})(\varphi(D,\bar{z},y))$ is consistent with $\tilde{U}$. From $(\exists y<\underline{a})(\exists \bar{z})\varphi(x,\bar{z},y) \in \exists\,\Phi\,(\underline{A})$ we obtain

$$\mathfrak{A} \models (\check{\exists}\, x)(\exists y<\underline{a})(Tr(x) \& (\exists \bar{z})\varphi(x,\bar{z},y)).$$

We have $\mathfrak{A} \models E((\exists \bar{z})\varphi(x,\bar{z},y))$ and, consequently,

$$\mathfrak{A} \models (\exists y<\underline{a})(\check{\exists}\, x)(Tr(x) \& (\exists \bar{z})\varphi(x,\bar{z},y)).$$

Let $b \in A$ be such that $\mathfrak{A} \models \underline{b}<\underline{a} \& (\check{\exists}\, x)(Tr(x) \& (\exists \bar{z})\varphi(x,\bar{z},\underline{b}))$. Following the property (3) of $\tilde{U}$ we obtain the model of $\tilde{U}$ in which $\varphi(D,\bar{c},\underline{b}) \& e = \underline{b}$ holds. Thus, (*) is proved.

We prove (ii). Let $\mathfrak{A} \subset_{\forall\exists\Phi} \mathfrak{B}$. Evidently $\mathfrak{B} \models \tilde{U}$. If $e \in A$, $\varphi(x,y) \in \exists\,\Phi\,(\underline{A})$ and $\mathfrak{A} \models (\check{\exists}\, x)(\exists y<\underline{e})(Tr(x) \& \varphi(x,y))$, then $\mathfrak{B} \models (\check{\exists}\, x)(\exists y<\underline{e}))(Tr(x) \& \varphi(x,y))$. Let $d \in B - A$ be such that $A < d$. There exists a $c \in B$ such that $\mathfrak{B} \models \underline{d}<\underline{c} \& Tr(\underline{c}) \& (\exists y<\underline{e})\varphi(\underline{c},y)$. Let $b \in B$ be such that $\mathfrak{B} \models \varphi(\underline{c},\underline{b}) \& \underline{b}<\underline{e}$. Evidently $b \in A$. For each $a \in A$ we have $a \overset{\mathfrak{B}}{<} c$ and, consequently, $\mathfrak{B} \models (\exists x)(\underline{a}<x \& Tr(x) \& \varphi(x,\underline{b})$. Thus, for each $a \in A$ we have $\mathfrak{A} \models (\exists x)(\underline{a}<x \& Tr(x) \& \varphi(x,\underline{b})$.

One implication of (iii) follows from (i) and the order one can be proved analogously as (ii).

Remark. In the part (i) of the proof we used only the following property of Φ : if $\varphi \in \Phi$, ψ is atomic then $\varphi \& \psi \in \Phi$.

§ 2. Π_n <u>end-extension theorem for countable structures</u>. We work with countable language L with bin. predicate $<$. The main theorem is formulated in 2.4.

2.0. For $\varphi(x,y) \in Fm(L)$ we denote by $H(\varphi(x,y))$ the general closure of the formula

$$(\forall u)((\forall x< u)(\exists y)\varphi \longrightarrow (\exists v)(\forall x< u)(\exists y< v)\varphi)$$

where u,v do not occur in φ . $H(\varphi)$ designates the $H(\varphi(x,y))$ with some x,y.

We put $$U^{(\Gamma)} = U \cup H(\Gamma)$$

where $\Gamma \subseteq Fm(L)$. We write U^n (U^{ω} resp.) instead of $U^{(\Pi_n)}$ ($U^{(Fm)}$ resp.).

2.1. Lemma. For $n \geq 0$ we have $\dot{C}lq(\Pi^{U^n}_{n+1})$.

Proof. By induction on n. For n = 0. If $\varphi \in \Sigma_1$ there is a $\psi \in \Delta_0$ such that $U \vdash \varphi \equiv (\exists y)\psi$. (See 1.0.) We have

$$U^0 \vdash (\forall x< u)\varphi \equiv (\forall x<u)(\exists y)\psi \equiv (\exists v)(\forall x< u)(\exists y \vee v)\psi$$

and consequently $(\forall x< u)\varphi \in \Sigma^{U^0}_1$. The formula $(\exists x< u)\varphi \in \Sigma^{U^0}_1$ follows immediately. Suppose the proposition is true for some n. For $\varphi \in \Sigma_{n+2}$ we have some $\psi \in \Pi_{n+1}$ such that $U^n \vdash \varphi \equiv (\exists y)\psi$. This follows from the induction hypothesis. Thus,

$$U^{n+1} \vdash (\forall x< u)\varphi \equiv (\forall x<u)(\exists y)\psi \equiv (\exists v)(\forall x<u)(\exists y< v)\psi.$$

From this and from the induction hypothesis we have $(\forall x < u)\varphi \in \Sigma^{U^{n+1}}_{n+2}$. Now $(\exists x<u)\varphi \in \Sigma^{U^{n+1}}_{n+2}$ immediately follows.

2.2. Lemma. For $n \geq 0$, U^n is equivalent to $U \cup E(\Sigma_n)$.

Proof. (a) $U \cup E(\Sigma_n) \prec U^n$. Let $\varphi(x,y) \in \Sigma_n$ be such that $(\forall y< u)(\exists z)(\forall x > z)(Tr(x) \longrightarrow \neg \varphi(x,y))$. We have $(\forall x > z)(Tr(x) \longrightarrow \neg \varphi(x,y)) \in \Pi_n$. By using $H(\Pi_n)$ we obtain $(\exists v)(\forall y< u)(\exists z< v)(\forall x > z)(Tr(x) \longrightarrow \neg \varphi(x,y))$. For $v< x$ and $Tr(x)$ we have $z< v \longrightarrow z \prec x$. Thus,

$(\exists v)(\forall y< u)(\forall x>v)(Tr(x)\longrightarrow \neg \varphi(x,y))$ holds and, consequently, $U^n \vdash (\forall y< u)(\exists z)(\forall x>z)(Tr(x)\longrightarrow \neg\varphi(x,y))\longrightarrow$ $\longrightarrow (\exists v)(\forall y< u)(\forall x>v)(Tr(x)\longrightarrow \neg\varphi(x,y))$. The last formula is equivalent to $E(\varphi(x,y))$.

(b) $U^n \prec U\cup E(\Sigma_n)$. For $\psi \in \Pi_k$ we denote by $\overline{\psi}(x,y)$ the formula $(\exists y_1< y)\,\psi(x,y_1)\,\&\,Tr(y)$. If

(1) $$(\forall x<u)(\exists y)\,\psi(x,y)$$

then

(2) $$(\forall x< u)(\exists z)(\forall y>z)(Tr(y)\longrightarrow \overline{\psi}(x,y))$$

We prove the statement (b) by induction.
For $n = k = 0$. We have $\overline{\psi} \in \Delta_0$. The formula

$$(\exists z)(\forall y>z)(\forall x<u)(Tr(y)\longrightarrow \overline{\psi}(x,y))$$

follows from (1), (2) and $E(\neg\overline{\psi})$. Thus, there is a v such that $Tr(v)$ and $(\forall x<u)\,\overline{\psi}(x,v)$, and consequently $(\forall x < u)(\exists y< v)\,\psi(x,y)$. Suppose (b) is true for some n. We put $k = n + 1$. By the induction hypothesis, $\overline{\psi} \in \Pi^{U^n}_{n+1}$ and $U^n \prec U\cup E(\Sigma_{n+1})$. From (1), $E(\neg\overline{\psi})$ we obtain a v such that $(\forall x<u)(\exists y<v)\,\psi(x,y)$ analogously as in the case $n = 0$.

2.3. <u>Lemma</u>. If $n\geq 0$ then $\mathfrak{A} \models U$ and $EE_{n+1}(\mathfrak{A})$ implies $\mathfrak{A} \models U^n$.

Proof. Let $\mathfrak{B}$ be such that $\mathfrak{A} \subset_{n+1} \mathfrak{B}$. Let $d\in B - A$ be such that $A< d$. We prove the lemma by induction on n. For $n = 0$: Let $\psi(x,y) \in \Delta_0(\underline{A})$ and let $a\in A$ be such that $\underline{\mathfrak{A}} \models (\forall x< \underline{a})(\exists y)\psi$. We have $\underline{\mathfrak{B}} \models (\forall x<\underline{a})(\exists y<\underline{d})\psi$. Thus, $\underline{\mathfrak{B}} \models (\exists v)(\forall x<\underline{a})(\exists y< v)\psi$, and consequently $\underline{\mathfrak{A}} \models (\exists v)(\forall x<u)(\exists y<v)\psi$.

For $n = 1$: If $\varphi(x,y)\in \Pi_1(\underline{A})$ then there is a $\psi \in \Delta_0(\underline{A})$ such that $U\vdash \varphi \equiv (\forall t)\psi$. We have $\mathfrak{B} \models U$. If $\underline{\mathfrak{A}} \models (\forall x< \underline{a})$

$(\exists y)\varphi(x,y)$ for some $a \in A$ then $\underline{\mathfrak{B}} \models (\forall x < \underline{a})(\exists y < \underline{d})\varphi$. Let us suppose

$(*)$ $\qquad \underline{\mathfrak{A}} \models (\forall v)(\exists x < \underline{a})(\forall y < v)\neg\varphi(x,y)$

Thus $\underline{\mathfrak{A}} \models (\forall v)(\exists x < \underline{a})(\forall y < v)(\exists t)\neg\psi$ holds. By using $H(\neg\psi)$ we have $\underline{\mathfrak{A}} \models (\forall y < v)(\exists t)\neg\psi \equiv (\exists w)(\forall y < v)(\exists t < w)\neg\psi$. Thus, $(*)$ implies $\underline{\mathfrak{A}} \models (\forall v)(\exists w)(\exists x < \underline{a})(\forall y < v)(\exists t < w)\neg\psi$. The last formula is from $\Pi_2(\underline{A})$, and consequently it holds in $\underline{\mathfrak{B}}$. For d we have $\underline{\mathfrak{B}} \models (\exists w)(\exists x < a)(\forall y < \underline{d})(\exists t < w)\neg\psi$. Thus, $\underline{\mathfrak{B}} \models (\exists x < \underline{a})(\forall y < \underline{d})\neg\varphi$, which is a contradiction. Now $\mathfrak{A} \models H(\varphi)$, and $\mathfrak{A} \models H(\Pi_1)$ is proved.

For n = 2: First, we prove $\mathfrak{B} \models H(\Delta_0)$. For $\varphi \in \Delta_0$ we have $H(\varphi) \in \Pi_3$. From this, $\mathfrak{A} \models H(\varphi)$ and $\mathfrak{A} \subset_3 \mathfrak{B}$ we obtain $\mathfrak{B} \models H(\Delta_0)$. We prove $\mathfrak{A} \models H(\Pi_2)$. If $\mathfrak{A} \models \Pi_2(\underline{A})$ then there is a $\psi \in \Sigma_1(\underline{A})$ such that $U^0(\underline{A}) \vdash \varphi \equiv (\forall t)\psi$. This follows from 2.1. We have $\mathfrak{A} \models U^0$ and $\mathfrak{B} \models U^0$. If $\underline{\mathfrak{A}} \models (\forall x < \underline{a})(\exists y)\varphi$ then $\mathfrak{B} \models (\forall z)(\forall x < \underline{a})(\exists y < \underline{d})(\forall t < z)\psi$. Thus, $\underline{\mathfrak{B}} \models (\exists v)(\forall z)(\forall x < \underline{a})(\exists y < v)(\forall t < z)\psi$ holds. We have $(\forall x < \underline{a})(\exists y < v)(\forall t < z)\psi \in \Sigma_1^{U^0(\underline{A})}(\underline{A})$, and $\mathfrak{A} \models (\exists v)(\forall z)(\forall x > \underline{a})(\exists y < v)(\forall t < z)\psi$. From this we obtain $\underline{\mathfrak{A}} \models (\exists v)(\forall x < \underline{a})(\forall z)(\exists y < v)(\forall t < z)\psi$. By using $\underline{\mathfrak{A}} \models E((\forall t < z)\psi)$ (this follows from $\mathfrak{A} \models H(\Pi_1)$ and 2.2) we have $\underline{\mathfrak{A}} \models (\forall z)(\exists y < v)(\forall t < z)\psi \rightarrow (\exists y < v)(\forall t)\psi$. Finally, $\underline{\mathfrak{A}} \models (\exists v)(\forall x < \underline{a})(\exists y < v)\varphi$.

For $n \geq 2$ we prove:

$(**)$ $\quad \mathfrak{A} \subset_{n+1} \mathfrak{B}$ implies $\underline{\mathfrak{A}} \models U^n$ and $\mathfrak{B} \models U^{n-2}$

By induction. For n = 2 the statement $(**)$ holds. Suppose $(**)$ is true for n. Let $\mathfrak{B}$ be such that $\mathfrak{A} \subset_{n+2} \mathfrak{B}$.

First, we prove $\mathcal{L} \models H(\Pi_{n-1})$. If $\varphi \in \Pi_{n-1}$ then $H(\varphi) \in \Pi_{n+2}^{U^{n-2}}$ holds. By using the induction hypothesis ($\mathcal{L} \models U^{n-2}$, $\mathfrak{A} \models H(\Pi_{n-1})$) we obtain $\mathcal{L} \models H(\Pi_{n-1})$. Thus, $\mathcal{L} \models U^{n-1}$ is proved. We shall prove $\mathfrak{A} \models H(\Pi_{n+1})$. If $\varphi \in \Pi_{n+1}(\underline{A})$ then there is a $\psi \in \Sigma_n(\underline{A})$ such that $U^{n-1}(\underline{A}) \vdash \varphi \equiv (\forall t)\psi$. If $\underline{\mathfrak{A}} \models (\forall x<\underline{a})(\exists y)\varphi(x,y)$ then $\underline{\mathcal{L}} \models (\forall z)(\forall x<\underline{a})(\exists y<\underline{d})(\forall t<z)\psi$. Thus, $\underline{\mathcal{L}} \models (\exists v)(\forall z)(\forall x<\underline{a})(\exists y<v)(\forall t<z)\psi$ holds. We have $(\forall x<\underline{a})(\exists y<v)(\forall t<z)\psi \in \Sigma_n^{U^{n-1}(\underline{A})}(\underline{A})$, and consequently $\underline{\mathfrak{A}} \models (\exists v)(\forall z)(\forall x<\underline{a})(\exists y<v)(\forall t<z)\psi$. By using $\mathfrak{A} \models E(\psi)$ we prove as in the case $n = 2$ that $\underline{\mathfrak{A}} \models (\exists v)(\forall x<\underline{a})(\exists y<v)\varphi$ holds. The proof is finished.

2.4. Theorem. Let $\mathfrak{A}$ be a countable model of U.

(a) For $n \geq 1$ we have: $EE_{n+1}(\mathfrak{A})$ iff $\mathfrak{A} \models U^n$ iff $\mathfrak{A} \models U \cup E(\Sigma_n)$.

(b) $EE_\omega(\mathfrak{A})$ iff $\mathfrak{A} \models U^\omega$ iff $\mathfrak{A} \models U \cup E(Fm)$.

Proof. (a) Only the part that $\mathfrak{A} \models U^n$ implies $EE_{n+1}(\mathfrak{A})$ is not easy. For $n \geq 1$ we put $\sigma_n^0 = \{\varphi \in \Sigma_n(\Pi_n$ resp.); there is a $\psi \in \Pi_n$ (Σ_n resp.) such that $U^n \vdash \varphi \equiv \psi\}$

$$\sigma_n^{m+1} = \{\varphi \,\&\, \psi ;\, \varphi, \psi \in \sigma_n^m\} \cup \sigma_n^m$$

$$\sigma_n = \bigcup_m \sigma_n^m.$$

Evidently, σ_n is closed under $\&$, $\neg$, subformulas and contains all atomic formulas. If $\varphi \in \sigma_n$ then there is a $\varphi_1 \in \Pi_n$ and a $\varphi_2 \in \Sigma_n$ such that $U^n \vdash \varphi \equiv \varphi_1 \,\&\, \varphi \equiv \varphi_2$. Thus, $U^n \vdash E(\exists \sigma_n)$. By using 1.4 we obtain a $\mathcal{L}$ such that

$\mathfrak{A} \prec_{\forall\exists\sigma_n} \mathfrak{B}$. We have $\Sigma_{n-1} \subseteq \sigma_n^0$ and, consequently, $\Pi_n \subseteq \subseteq \forall\exists \Sigma_{n-1} \subseteq \forall\exists \sigma_n$. This implies $\mathfrak{A} \prec_n \mathfrak{B}$.

(b) follows immediately from 1.4 and 2.2.

2.5. Proposition. For $n \geq 0$ we have $U^{(\Pi_n)} \equiv U^{(\Sigma_{n+1})}$.

Proof. For $\varphi \in \Sigma_{n+1}$ we have a $\psi \in \Pi_n$ such that $U^n \vdash \varphi \equiv (\exists t)\psi$. Assume $(\forall x < u)(\exists y)\varphi(x,y)$ is provable in U^n. Thus, $U^n \vdash (\forall x < u)(\exists z)(\exists y,t < z)\psi$. By 2.1, we have $(\exists y,t < z)\psi \in \Pi_n^{U^n}$. From this we obtain $U^n \vdash (\exists v)(\forall x < u)(\exists z < v)(\exists y,t < z)\psi \;\&\; Tr(v)$. Consequently, $U^n \vdash (\exists v)(\forall x < u)(\exists y < v)(\exists t)\psi$, i.e. $U^n \vdash (\exists v)(\forall x < u)(\exists y < v)\varphi$.

2.6. Remark. Every countable model of ZF (Zermelo-Fraenkel set theory) has an elementary end-extension. This follows immediately from 2.4 (b).

§ 3. Relations between the schemas $H(\Pi_n)$ and $Ind(\Pi_n)$.

3.0. For $\varphi(x)$ we denote by $Ind(\varphi(x))$ the general closure of the formula

$$(\forall x)((\forall y < x)\varphi(y) \rightarrow \varphi(x)) \rightarrow (\forall x)\varphi(x)$$

and by $Min(\varphi(x))$ the general closure of the formula

$$(\exists x)\varphi(x) \rightarrow (\exists x)(\varphi(x) \;\&\; (\forall y < x)\neg\varphi(y)).$$

Evidently, $Ind(\varphi(x)) \equiv Min(\neg\varphi(x))$.

3.1. Theorem. For each $n \geq 0$ we have

$$U^n \cup Min(\Delta_0) \vdash Min(\Pi_n \cup \Sigma_n).$$

Proof. By induction on n. The case n = 0 is trivial.

Suppose the statement is true for n. If $\psi_0 \in \Pi_n$ (Σ_n resp.) there is a $\psi \in \Pi_n$ (Σ_n resp.) such that $U^n \vdash (\exists \bar{x}) \psi_0 \equiv (\exists x) \psi$ ($U^n \vdash (\forall \bar{x}) \psi_0 \equiv (\forall x) \psi$ resp.). Thus, if $\varphi \in \Pi_{n+1}$ (Σ_{n+1} resp.) then there is a $\psi \in \Sigma_n$ (Π_n resp.) such that $U^n \vdash (\forall x) \psi \equiv \varphi$ ($U^n \vdash \varphi \equiv (\exists x) \psi$ resp.).

(1) Let $\varphi \in \Sigma_{n+1}$ be such that φ has the form $(\exists y)\psi$ with some $\psi \in \Pi_n$. If $U^{n+1} \vdash (\exists x)\varphi(x)$ there is a u_1 such that $U^{n+1} \vdash Tr(u_1) \& (\exists x < u_1)\varphi(x)$. We put $\hat{\psi}(x,y) \equiv \psi(x,y) \vee (\neg (\exists z)\psi(x,z))$. We have $\hat{\psi} \in \Pi_{n+1}$ and $U^{n+1} \vdash (\forall x < u_1)(\exists y)\hat{\psi}(x,y)$. By using $H(\Pi_{n+1})$ we obtain a v such that $(\forall x < u_1)(\exists y < v)\hat{\psi}(x,y)$. Let $\varphi_v(x)$ be the formula $(\exists y < v)\psi$. We have $\varphi_v(x) \in \Pi^{U^n}_n$. By the induction hypothesis there is a u_0 such that $\varphi_v(u_0) \& u_0 < u_1 \& (\forall u < u_0)\neg \varphi_v(u)$. $\varphi(u_0)$ follows evidently. For $u < u_0$ we have $\neg (\exists y < v)\psi(u,y)$. Suppose $(\exists y)\psi(u,y)$. Then $(\exists y < v)\psi(u,y)$ follows from $(\exists y < v)\hat{\psi}(u,y)$, which is a contradiction. Thus $u < u_0 \longrightarrow \neg \varphi(u)$.

(2) Let $\varphi \in \Pi_{n+1}$ be such that φ has the form $(\forall y)\psi$ with some $\psi \in \Sigma_n$. We suppose $(\exists x < u_1)\varphi(x) \& Tr(u_1)$. We put

$$\hat{\psi}(x,y) \equiv \neg \psi(x,y) \vee ((\forall z)\psi(x,z));$$

$\hat{\psi} \in \Pi_{n+1}$ follows evidently. Furthermore, we have $(\forall x < u_1)(\exists y)\hat{\psi}(x,y)$. We choose a v such that $(\forall x < u_1)(\exists y < v)\hat{\psi}$ by using $H(\hat{\psi})$ and put $\varphi_v(x) \equiv (\forall y < v)\psi$. We have $\varphi_v \in \Sigma^{U^n}_n$ and in the same way as in (1) we finish the proof.

3.2. Let L be a countable language with binary predicate $<$ and a constant 0. In L consider the following theory S:

$<$ is a non symmetric linear ordering with the least element 0 and without the last element satisfying moreover,

$x \neq 0 \longrightarrow (\exists y)(y < x \,\&\, (\forall z < x)(z < y \vee z = y))$.

Evidently S has a Π_2 system of axioms and $U \prec S$. We define S^{Γ}, S^n analogously as U^{Γ}, U^n in 2.0.

3.3. Proposition. $S^n \prec S \cup \mathrm{Min}(\Pi_{n+1})$ holds for $n \geq 0$.

Proof by induction on n: For n = 0. Let $\varphi \in \Delta_0$ be such that

$$(\forall x < \tilde{u})(\exists y)\varphi(x,y) \,\&\, \neg(\exists v)(\forall x < \tilde{u})(\exists y < v)\varphi(x,y).$$

Let $u_0 \leq \tilde{u}$ be the least element t such that $\neg(\exists v)(\forall x < t)(\exists y < v)\varphi$. We have $0 < u_0$. We take a u_1 such that $u < u_0 \longrightarrow u < u_1 \vee u = u_1$ (the predecessor of u_0). There is a v_1 such that $(\forall x < u_1)(\exists y < v_1)\varphi$. We have $u_1 < \tilde{u}$. Thus, there is a y_1 such that $\varphi(u_1,y_1)$. Let v_0 be such that $v_1 < v_0$ and $y_1 < v_0$, Thus, we have $(\forall x < u_0)(\exists y < v_0)\varphi$, which is a contradiction.

Suppose the proposition is true for n. Let $\varphi \in \Pi_n$ be such that

$$(\forall x < \tilde{u})(\exists y)\varphi(x,y) \,\&\, \neg(\exists v)(\forall x < \tilde{u})(\exists y < v)\varphi .$$

By using the induction hypothesis and 2.1 we obtain $\neg(\exists v)(\forall x < \tilde{u})(\exists y < v)\varphi \in \Pi_{n+1}^{S^{n-1}}$. Thus, there is a u_0 such that u_0 is the least t for $\neg(\exists v)(\forall x < t)(\exists y < v)\varphi$. The proof can be finished as in the case n = 0.

3.4. We put for $T \vdash U$ ($T \vdash S$ resp.) and $\Gamma \subseteq \mathrm{Fm}(L)$:

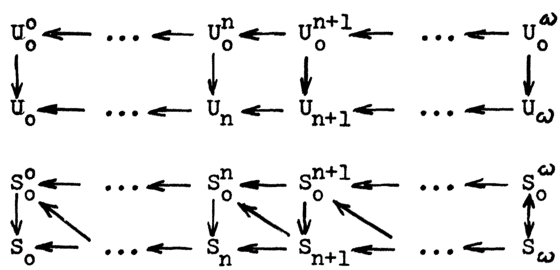
$$T_{(\Gamma)} = T \cup \mathrm{Min}(\Gamma).$$

We denote $T_{(\Pi_n \cup \Sigma_n)}$ ($T_{(Fm)}$ resp.) by T_n (T_ω resp.).

From 3.0 - 3.3 and 2.5 we deduce for $n \geq 0$

$U^n \equiv U^{(\Sigma_{n+1})}$, $U_n \prec U_o^n$ and $S^n \prec S_{n+1}$.

We can visualize these results in the following diagrams (where $T \longrightarrow S$ indicates that $T \succ S$)



3.5. Proposition. If $n \geq 0$, $\mathfrak{A} \models S_{(\Pi_n)}$ and $\mathfrak{A} \subset_{n+2} \mathfrak{B}$, then $\mathfrak{B} \models S_{(\Pi_n)}$.

Proof by induction on n: For n = 0. $\varphi \in \Delta_o$ implies $\mathrm{Min}(\varphi) \in \Pi_2$. Thus, $\mathfrak{B} \models S_o$. Assume the statement is valid for n. Let $\mathfrak{A} \subset_{n+3} \mathfrak{B}$ be such that $\mathfrak{A} \models S_{(\Pi_{n+1})}$. We deduce $\mathfrak{B} \models S^{n-1}$ by the induction hypothesis. For $\varphi \in \Pi_n$ we have $H(\varphi) \in \Pi^{S^{n-1}}_{n+3}$. Thus, $\mathfrak{B} \models H(\varphi)$ and, consequently, $\mathfrak{B} \models S^n$. If $\varphi \in \Pi_{n+1}$ then $\mathrm{Min}(\varphi) \in \Pi^{S_n}_{n+3}$. Now, $\mathfrak{B} \models \mathrm{Min}(\varphi)$ follows immediately.

§ 4. Relations between end-extensions and the induction schema.

4.0. From the results of the previous sections we obtain immediately

Theorem.

(a) If $n \geq 0$, $\mathfrak{A} \models U_0$ and $EE_{n+1}(\mathfrak{A})$ then $\mathfrak{A} \models U_n$.

(a1) If $\mathfrak{A} \models U_0$ and $\mathfrak{A}$ has an elementary end-extension then $\mathfrak{A} \models U_\omega$.

(b) If $n \geq 1$, $\mathfrak{A} \models S_{(\Pi_{n+1})}$ and $\mathfrak{A}$ is countable then $EE_{n+1}(\mathfrak{A})$.

(b1) If $\mathfrak{A} \models S_\omega$ and $\mathfrak{A}$ is countable then there is an elementary end-extension of $\mathfrak{A}$.

4.1. Definition. The model $\mathfrak{A}$ of U is said to be $\aleph_1$-like if

(1) the cardinality of A is $\aleph_1$,

(2) the set $\{a \in A;\ a \overset{\mathfrak{A}}{<} b\}$ is countable for all $b \in A$.

Proposition. Let $\mathfrak{A}$ be an $\aleph_1$-like model of U_0. Then $\mathfrak{A} \models U_\omega$.

Proof. We have $\mathfrak{A} \models U^\omega$ and consequently the proposition follows by using 3.1.

Proposition. For a countable model $\mathfrak{A}$ of S_0 the following statements are equivalent:

(i) $\mathfrak{A}$ has an elementary end-extension

(ii) $\mathfrak{A} \models S_\omega$

(iii) $\mathfrak{A} \models S^\omega$

(iv) $\mathfrak{A} \models E(Fm)$

(v) $\mathfrak{A}$ has an $\aleph_1$-like elementary end-extension.

Proof. Only implication (iv) $\longrightarrow$ (v) needs an explicit proof. Assume (iv). We can construct the end-elementary ω_1 chain with the first member $\mathfrak{A}$. (Use 2.4 (b).) The required $\aleph_1$-like elementary end-extension is the limit of this chain.

4.2. Let L be a usual language of Peano arithmetic. The arithmetic A has the closures of the following formulas for axioms:

$$x + 0 = x,\ x + y = y + x,\ x + (y + z) = (x + y) + z,$$
$$x + y' = (x + y)',$$
$$x.0 = 0,\ x.y = y.x,\ x.(y.z) = (x.y).z,\ x.y^1 = x.y + x,$$
$$x.(y + z) = x.y + x.z,$$
$$\neg(x < x),\ x < y \,\&\, y < z \longrightarrow x < z,\ x < y \vee x = y \vee y < x,\ x < y' \longleftrightarrow$$
$$\longleftrightarrow x < y \vee x = y,\ 0 = x \vee 0 < x,\ 0 < x \longrightarrow (\exists y < x)(y' = x),\ x < x'.$$

All axioms of A are Π_1 formulas, and $A \vdash x \neq 0 \longrightarrow$ $\longrightarrow (\exists y)(\forall z < x)(z < y \vee z = y)$. Thus, $S \prec A$. We denote A_ω by P (as usual).

<u>Proposition</u>. (1) $P \equiv A_0^\omega \equiv A_0 \cup E(Fm)$.

(2) Let $\mathfrak{A}$ be a countable model of A_0.

$\mathfrak{A} \models P$ iff $\mathfrak{A}$ has an elementary end-extension.

(3) If $n \geq 1$ and $\mathfrak{A}$ is a countable model of $A(\Pi_{n+1})$ then $EE_{n+1}(\mathfrak{A})$.

(4) If $n \geq 0$, $\mathfrak{A} \models A_0$ and $EE_{n+1}(\mathfrak{A})$, then $\mathfrak{A} \models A_n$.

(5) If $n \geq 1$, $\mathfrak{A}$ is countable and $\mathfrak{A} \models A$, then $EE_{n+1}(\mathfrak{A})$ iff $\mathfrak{A} \models A^n$ iff $\mathfrak{A} \models A \cup E(\Sigma_n)$.

R e f e r e n c e s

[1] KIESLER H.J.: Forcing and the Omitting Types Theorem

[2] SHOENFIELD J.R.: Mathematical Logic, Addison Wesley (1967)

Matematický ústav
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 22.12. 1977)