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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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ON THE PERRON-FROBENIUS THEORY FOR SETS OF POSITIVE OPERATORS

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Abstract: In this note the Perron-Frobenius theory for certain sets of positive integral operators is developed. The asymptotic behaviour of iterated operators M^n is studied, and the existence of a global subinvariant function for the corresponding maximal Perron eigenvalue is established. Some consequences of this results are presented.

Key words: Positive integral operator, Perron-Frobenius eigenvalue, subinvariant function, compact sets of operators.

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1. In paper [3], in connection with a dynamic programming problem, Perron-Frobenius theory for sets of non-negative operators given by finite or countably infinite matrices is developed. In this paper we generalize some results on sets of integral operators. We are concerned mainly with the existence of a global Perron eigenvalue and a corresponding subinvariant function. In a wide class of controlled Markov processes of the diffusion type to each stationary control strategy there corresponds a non-negative operator (potential). The assumptions of our theorems

are fulfilled for sets of potentials for a subclass of such processes. Thus, this note may be useful for the study of the properties of controlled Markov processes.

Let us give a simple example: Diffusion on $[0,1)$ with generating differential operator $d^2/dx^2 + u(x)\, d/dx$, with reflection in 0 and absorption in 1, has the potential

$$(Mf)(x) = \int_0^1 Q(y) \int_{x \vee y}^1 Q(s)^{-1} ds\ f(y) dy,$$

where $Q(y) = \exp(\int_0^y u(s) ds)$, $x \vee y = \max(x,y)$.

If we interpret here $u(x)$ as a stationary control strategy from the class of all Borel measurable functions on $[0,1)$ bounded by a given constant, we obtain a set of potentials fulfilling our assumptions.

2. Let X be a Borel set in R^d and let V be a non-negative σ-finite measure on its Borel subsets. Let $m(x,y)$ be a measurable non-negative function on $X \times X$. We define operator M by

$$(1) \qquad (Mf)(x) = \int_X m(x,y) f(y) dV(y).$$

We shall say that operator M in (1) has a density m, if

$$\| m \| = \sup_{x \in X} \int_X m(x,y) dV(y) < \infty .$$

If we introduce the multiplication of densities by

$$(2) \qquad m_1 \circ m_2(x,y) = \int m_1(x,z) m_2(z,y) dV(z),$$

then the iterated operator M^n has a density $m_n =$
$= \underbrace{m \circ m \circ \dots \circ m}_{n}$.

Definition 1. A function μ not identically zero is called right eigenfunction of M corresponding to eigenvalue φ if

(3) $\varphi\mu(x) = \int m(x,y)\mu(y)dV(y)$ for all $x \in X$.

Similarly we define the left eigenfunction ν not equal zero a.e., namely by the relation

(4) $\varphi\nu(y) = \int \nu(x)m(x,y)dV(x)$ for V-almost all $y \in X$.

We recall that function f is said uniformly positive if there exists a constant $c > 0$ with $f \geq c$ everywhere.

For our purposes we need the following

Theorem 1. Suppose $0 < V(X) < \infty$, and let there exist a natural number n such that m_n is a uniformly positive bounded function. Then operator M with density m has a positive eigenvalue $\varphi > 0$ in absolute value greater than any other. The corresponding eigenfunction μ is bounded and uniformly positive and the same holds about ν a.e. Moreover μ and ν are unique up to a multiplicative constant. Let $\int \mu\nu\, dV = 1$, then

(5) $m_n(x,y) = \varphi^n\mu(x)\nu(y)(1 + O(\Delta^n)),\ n \longrightarrow \infty,\ 0 < \Delta < 1,$

where Δ is independent of x, y.

The proof with a slight modification can be found in [2].

Definition 2. For functions f, g on X we write $f \succ g$ if $f \geq g$ and $f > g$ on a set of positive measure.

Like in [2] it can be also proved that for the maximal eigenvalue ϱ holds

$$(6) \qquad \varrho = \sup \{ \sigma > 0 : \exists\, f \gneq 0 \text{ bounded: } Mf \geq \sigma f \} .$$

<u>Lemma 1</u>. If the conditions of Theorem 1 are satisfied then

$$(7) \qquad \varrho = \sup_{f \gneq 0} \inf_{x} \int m(x,y)f(y)dV(y)/f(x),$$

where $Mf(x)/f(x) = \infty$ for $f(x) = 0$.

Proof. We denote for arbitrary bounded function $f \gneq 0$

$$r_1 = r_1(f) = \inf_{x} Mf(x)/f(x),$$

$$(8) \qquad r_2 = r_2(f) = \sup \{ \sigma > 0 : Mf \geq \sigma f \} .$$

Obviously, $\varrho = \sup r_2(f)$. Now it is $r_1 \geq r_2$. If it were $r_1 < r_2$, then by (8) it would exist a $\sigma > r_1$ such that $Mf \geq \sigma f$. But for $\sigma > r_1$ there exists $x \in X$ such that $Mf(x)/f(x) < \sigma$. This is a contradiction. On the other hand, by (8) is $r_1(f)f \leq Mf$, so we have $r_1(f) \leq r_2(f)$ for every $f \gneq 0$. This all gives our assertion.

3. Let operator M have density m. The following lemma gives us an information about the asymptotic growth of densities m_n as $n \longrightarrow \infty$.

<u>Lemma 2</u>. If the conditions of Theorem 1 are satisfied then the infinite series $\sum_{n=1}^{\infty} m_n(x,y)z^n$ have the same radius of convergence $R = 1/\varrho$ for every $x \in X$ for almost all $y \in X$.

Proof. For the radius of convergence R of the series

$\Sigma\, m_n(x,y)z^n$ we have from (5) for every x for almost all y $1/R = \limsup\limits_{n\to\infty} \sqrt[n]{m_n(x,y)} = \limsup\limits_{n\to\infty} \varphi \sqrt[n]{\mu(x)\nu(y)}$ $\sqrt[n]{1 + O(\Delta^n)} = \varphi$.

The following notion plays an important role in the sequel.

Definition 3. Let function f be positive and finite on a set of positive measure V.,We call f β-subinvariant for operator M with density m if

$$(9) \qquad \beta Mf \le f, \; 0 \le \beta < \infty \; .$$

Iterating this relation we get for β-subinvariant function f $\beta^n M^n f \le f$. If M is an operator with density m and for some natural n is $0 < a \le m_n \le b < \infty$ then we have for $\beta > 0$ $0 < \beta^n a \int f(y)dV(y) \le \beta^n M^n f \le f$, hence f is uniformly positive. On the other hand for x such that $f(x) < \infty$ we have $\beta^n \int m_n(x,y)f(y)dV(y) \le f(x)$. Hence, f is finite a.e.

Lemma 3. If R(x,y) is the radius of convergence of the series $\sum_{n=1}^{\infty} m_n(x,y)z^n$, then for $\beta > \sup_{y \in X} R(x,y)$ there exists no β-subinvariant function.

Proof. Suppose that there exists such a function then we have for $\beta > \bar{\beta} > \sup R(x,y)$:

$$\int_X \sum_{n=1}^{\infty} m_n(x,y)\bar{\beta}^n f(y)dV(y) = \sum_{n=1}^{\infty} \bar{\beta}^n \int_X m_n(x,y)f(y)dV(y) \le \sum_{n=1}^{\infty} f(x)(\bar{\beta}/\beta)^n < \infty$$

. This contradicts the inequality $\bar{\beta} > \sup R(x,y)$.

In the previous lemma we can take only $\beta > \sup_{y \in A} R(x,y)$

where $V(A) > 0$, and the proof proceeds in the same manner. Thus, we have

Lemma 4. If $R(x,y)$ is the radius of convergence of the series $\sum_{n=1}^{\infty} m_n(x,y)z^n$, $A \subset X$, $V(A) > 0$, then for $\beta > \sup_{y \in A} R(x,y)$ there exists no β-subinvariant function.

Corollary 1. If the conditions of Theorem 1 are satisfied and R is the common convergence radius of series $\sum m_n(x,y)z^n$, then for $\beta > R$ no β-subinvariant function can exist.

4. Up to now we have studied operators with densities in the case of finite measure V on X. Suppose for more general case that $X \subset R^d$ is σ-compact and V non-negative measure finite on compacts. Then there exist X_k compact such that $X = \bigcup_{k=1}^{\infty} X_k$, $V(X_k) < \infty$. We can choose X_k so that $X_k \subset \subset X_{k+1}$. V is a σ-finite measure on X. If we are given an operator M with density m, we can for any natural k define operator M_k with density $m^{(k)}$ where

$$m^{(k)}(x,y) = m(x,y),\ (x,y) \in X_k \times X_k, \quad = 0 \text{ otherwise.} \tag{10}$$

In this case we shall additionally suppose that the following conditions are satisfied:

(i) m is lower semicontinuous on $X \times X$,

(ii) $\forall \varepsilon > 0\ \exists k \in \mathbb{N} : \int_{y \notin X_k} m(x,y)dV(y) < \varepsilon$, $x \in X$,

(iii) $0 < m(x,y) \leq C_k < \infty$ for every x a.e. on X_k.

Theorem 2. Let $X \subset R^d$ be σ-compact and V non-negative measure finite on compacts, let M be an operator with density m which satisfies (i) - (iii). Let R_k denote the common radius of convergence of series $\sum_{n=1}^{\infty} m_n^{(k)}(x,y)z^n$ where we define

$$m_n^{(k)}(x,y) = \int_{X_k} m^{(k)}(x,z)m_{n-1}^{(k)}(z,y)dV(z).$$

Then for the radii of convergence $R(x,y)$ of the infinite series $\sum m_n(x,y)z^n$ holds for each x $R(x,y) = R$ for almost all y, $R = \lim_{k \to \infty} R_k$.

Proof. The operator M_k with density $m^{(k)}$ satisfies conditions of Theorem 1 because of compactness of X_k and because of lower semicontinuity of $m > 0$. The series $\sum m_n^{(k)}(x,y)z^n$ has the convergence radius $R_k = 1/\varphi_k$, where $\varphi_k > 0$ is the corresponding eigenvalue. We have for any $(x,y) \in X \times X$

$$m_n^{(k)}(x,y) \leq m_n^{(k+1)}(x,y) \leq m_n(x,y),$$

so that for the convergence radius $R(x,y)$ holds

$$R_k \geq R_{k+1} \geq R(x,y) \geq 0 \text{ and hence } R_k \downarrow R \geq R(x,y).$$

Let μ_k be the right eigenfunction corresponding to the eigenvalue φ_k by Theorem 1 such that $\sup_{x \in X_k} \mu_k(x) = 1$. We set $\mu_k = 0$ on the complement of X_k in X. Then from the relation

$$\mu_k(x) = R_k \int_{X_k} m^{(k)}(x,y) \mu_k(y)dV(y)$$

we get by Fatou's lemma

$$1 \geq \mu(x) = \liminf_{k \to \infty} \mu_k(x) \geq R \int_X m(x,y) \mu(y)dV(y),$$

i.e. $\mu \geq RM\mu$. Now we show that μ is positive on a set of positive measure. Let X_r be such that $V(X_r) > 0$, and

$$\int_{y \notin X_r} m(x,y)dV(y) < \frac{1}{4R_1}, \quad x \in X.$$

We can take x_k, $k \in \mathbb{N}$, such that

$$\frac{3}{4} \leq \mu_k(x_k) = R_k \int m(x_k,y)\ \mu_k(y)dV(y) <$$

$$R_k \int_{X_r} m(x_k,y)\ \mu_k(y)dV(y) + \frac{1}{4} R_k/R_1 \leq R_k C_r \int_{X_r} \mu_k(y)dV(y) + \frac{1}{4}.$$

Hence, $R_k \int_{X_r} \mu_k(y)dV(y) \geq \frac{1}{2C_r}$, and we have for $x \in X_r$, $m(x,y) \geq a_r > 0$ on $X_r \times X_r$, an inequality

$$\mu_k(x) \geq R_k \int_{X_r} m(x,y)\ \mu_k(y)dV(y) \geq \frac{a_r}{2C_r} > 0, \quad k \geq r.$$

We have shown that μ is an R-subinvariant function. $R(x,y)$ is obviously a measurable function on $X \times X$. Taking $\varepsilon > 0$ arbitrary and applying Lemma 4 we get $R(x,y) \geq R - \varepsilon$ for almost all $y \in X$. This gives $R(x,y) = R$ for each x for almost all y.

As an easy consequence of Lemma 4 we obtain

Corollary 2. If the conditions of Theorem 2 are satisfied then for $\beta > R$ there exists no β-subinvariant function.

Corollary 3. For operator M satisfying the conditions of Theorem 2 does exist an R-subinvariant function.

5. In this section we shall deal with the sets S of operators M with densities satisfying conditions of Theorem 1. The radius of convergence R corresponding to operator $M \in S$ by Lemma 2 will be denoted by R(M). Analogously we shall write $\varrho(M)$, $\mu(M)$, $\nu(M)$.

We define a distance in the set of densities by the relation

$$\| m - m' \| = \sup_x \int_X | m(x,y) - m'(x,y) | \, dV(y),$$

and denote by $\mathfrak{M}(X)$ the metric space thus obtained. The set of operators will be identified with the set of densities $S \subset \mathfrak{M}(X)$ which will be assumed to fulfil the following conditions :

(i) S is compact in $\mathfrak{M}(X)$,

(ii) for any measurable function $f \gneq 0$ (finite a.e.), and any $M, N \in S$ there exists $P \in S$ such that $Pf \geq Mf$, $Pf \geq Nf$.

In combination with condition (i) we shall use the following theorem about the continuity of the eigenvalue on S.

<u>Theorem 3</u>. Let there exist a natural n and positive numbers a, b such that for all $M \in S$ holds $0 < a \leq m_n(x,y) \leq$ $\leq b < \infty$. Then ϱ is a continuous function of m.

Proof. First assume n = 1. Take $M' \in S$ and denote as in (8) $r_2'(f) = \sup\{\sigma > 0 : M'f \geq \sigma f\}$. Let M be such that $\| m - m' \| < \delta$. Then for $g \gneq 0$ bounded

$$Mg(x) = M'g(x) + \int (m(x,y) - m'(x,y))g(y)dV(v) \geq$$

$$\geq M'g(x) - \delta \sup_x g(x).$$

Let $f \succ 0$ be bounded. We can norm f without loss of generality so that $\int f dV = 1$. Set $g(x) = M'f(x)$, i.e. $a \leq g \leq b$, and suppose $M'f \geq \sigma f$. Then we have $M'g(x) \geq \sigma M'f(x) = \sigma g(x)$, and hence $Mg(x) \geq \sigma g(x) - \delta b \geq g(x) (\sigma - \delta b/a)$. It is new sufficient to take $\delta > 0, \delta b/a < \varepsilon$, and to conclude that

$$\| m - m' \| < \delta \Longrightarrow [\forall f \succ 0 \; \exists g \succ 0 : M'f \geq \sigma f \longrightarrow Mg \geq$$
$$\geq (\sigma - \varepsilon) g] .$$

The last implication means that $r_2'(f) \leq r_2(g) + \varepsilon$. Consequently $\rho(M') \leq \rho(M) + \varepsilon$. Interchanging M and M' we get

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \| m - m' \| < \delta \Longrightarrow | \rho(M) - \rho(M') | < \varepsilon .$$

Suppose now that n is arbitrary. It is clear that ρ^n is the eigenvalue of M^n with density m_n. As above we conclude that ρ^n is a continuous function of m_n, and the same is true for ρ. Since m_n depends continuously on m, we obtain from here the continuity of ρ on S.

The compactness of S and the continuity $\rho(M)$ on S implies the existence of an operator $M^* \in S$ such that

$$\hat{R} = \min_{M \in S} R(M) = R(M^*).$$

It follows that $\hat{R} > 0$.

<u>Theorem 4</u>. $\hat{R}$ is the greatest among those numbers β for which there exists a function $f \succ 0$ finite a.e. such that

$$\text{(11)} \qquad \sup_{M \in S} \beta M f \leq f.$$

If $\hat{R} = R(M^*)$ where $M^* \in S$, then relation (11) holds a.e. for $\beta = \hat{R}$ if and only if $f = c\mu(M^*)$ a.e. where $c > 0$ is a constant.

Proof. For the existence of such f we must have $\beta \leq R(M)$ for an arbitrary $M \in S$, hence $\beta \leq \min R(M) = \hat{R}$. Let $f = c\mu(M^*)$ a.e. Suppose that

(12) $$\sup_{M \in S} \hat{R}Mf \leq f \quad \text{a.e.}$$

doesn't hold. Then there exists an operator $N \in S$ such that $\hat{R}Nf \not\leq f$. But $\hat{R}M^* f = f$ a.e., hence we can find, by Condition (ii), an operator $P \in S$ with density p for which we have $\hat{R}Pf \not\leq f$. Multiplying this inequality by $R(P)\nu(P)$ and integrating with respect to x we get

$$\hat{R} \ R(P) \iint \nu(P)(x)p(x,y)f(y)dV(y)dV(x) =$$

$$= \hat{R} \int \nu(P)(y)f(y)dV(y) > R(P) \int \nu(P)(x)f(x)dV(x).$$

This gives $\hat{R} > R(P)$, a contradiction to the definition of $\hat{R}$. Hence (12) is valid.

Suppose now that (12) holds for some function $f \not\leq 0$ finite a.e. In particular $\hat{R}M^* f \leq f$ a.e. If $\hat{R}M^* f = f$ a.e. were not true, then multiplying the above inequality by $\nu(M^*)$ and integrating with respect to x we get

$$\hat{R} \int \nu(M^*)(x)M^* f(x)dV(x) = \int \nu(M^*)(y)f(y)dV(y) <$$

$$< \int \nu(M^*)(x)f(x)dV(x).$$

This cannot hold. We can change f on a set of zero measure to obtain (11) with $\beta = \hat{R}$ everywhere in X. From the unicity of the eigenfunction corresponding to $\rho(M^*)$ we conclude that $f = c\mu(M^*)$ a.e.

6. In this section we shall study the set S of operators which satisfy the hypotheses of Section 4. S is supposed to fulfil Conditions (i) and (ii) of the previous section. We denote the corresponding convergence radius by R(M) for $M \in S$, and again we define $\hat{R} = \inf R(M)$. X_k, $R_k(M)$ etc. are defined as in Section 4.

Lemma 5. Let $\hat{R}_k = \inf_{M \in S} R_k(M)$. Then $\hat{R}_k \downarrow \hat{R}$.

Proof. We take M^* such that $\hat{R}_k = R_k(M^*)$. As in the proof of Theorem 2 we get

$$\hat{R} \leq R(M^*) \leq R_{k+1}(M^*) \leq R_k(M^*).$$

It follows, for any natural k, that

$$0 \leq \hat{R} \leq \hat{R}_{k+1} \leq \hat{R}_k, \text{ so that } \hat{R}_k \downarrow R^* \geq \hat{R}.$$

For any operator $M \in S$ there is $\hat{R}_k \leq R_k(M)$, so as $k \longrightarrow \infty$ we have $R^* \leq R(M)$. This gives $\hat{R} = \inf R(M) \geq R^* \geq \hat{R}$, and the proof is complete.

Now we shall prove an analogue of Theorem 4 of the previous section.

Theorem 5. $\hat{R}$ is the greatest among those numbers β for which there exists a function $f \succ 0$ finite a.e. such that

$$\beta Mf \leq f \text{ for all } M \in S. \tag{13}$$

Proof. Using Corollary 2 we obtain easily that such a function f satisfying (13) cannot exist for $\beta > \hat{R}$. Let us prove its existence for $\beta = \hat{R}$. The case $\hat{R} = 0$ is trivial. Let $\hat{R} > 0$. By Theorem 4 and by Theorem 1 there exists for any natural k a positive function f_k on X_k such that sup $f_k = 1$, and $\hat{R}_k M_k f_k \leq f_k$ for all $M \in S$, i.e.

$$\hat{R}_k \int m^{(k)}(x,y) f_k(y) dV(y) \leq f_k(x),$$

consequently, by Fatou's lemma, for $f = \liminf_{k \to \infty} f_k$ we have $\hat{R}Mf \leq f$ where we set $f_k = 0$ on the complement of X_k in X. Exactly in the same manner as in Theorem 2 we can show that f is positive on a set of positive measure. Thus we have found an $\hat{R}$-subinvariant function f for all $M \in S$.

7. Suppose now that S is a set of operators satisfying the conditions of Sections 5 or 6, and let $\{f_n\}_{n=0}^{\infty}$ be a sequence of functions defined by the recurrence

(14) $\qquad f_{n+1} = M_n f_n, \quad n = 0,1,\ldots, M_n \in S,$

where $f_0 \not\leq 0$ is an arbitrary measurable function finite a.e. We shall be interested in a certain maximal sequence which is obtained by the recurrence

(15) $\qquad \hat{f}_{n+1}(x) = \sup_{M \in S} M\hat{f}_n(x), \quad x \in X,$

where $0 \prec \hat{f}_0 \leq f$, and f is the function satisfying relation (11) with $\beta = \hat{R}$, existing by Theorems 4 or 5, respectively.

Under these assumptions, the following theorem holds, which gives us an information about the asymptotic behaviour of the sequence $\{\hat{f}_n\}$.

Theorem 6. The series $\sum_{n=0}^{\infty} \hat{f}_n(x) z^n$, $x \in X$, all have common radius of convergence $\hat{R}$.

Proof. Suppose $\hat{R} > 0$, by induction it follows that $\hat{f}_n \leq \hat{R}^{-n} f$, $n \geq 0$. Thus for $0 < z < \hat{R}$ we have

$$\sum_{n=0} \hat{f}_n(x) z^n \leq f(x) \sum_{n=0}^{\infty} \hat{R}^{-n} z^n = f(x)(1 - z/\hat{R})^{-1} < \infty .$$

This implies that every series has convergence radius not less than $\hat{R}$. This is trivially true if $\hat{R} = 0$.
Now suppose that some series $\sum \hat{f}_n(x) z^n$ has convergence radius $R(x) > \hat{R}$. We take $M^* \in S$ such that

$$\hat{R} \leq R(M^*) < R(x), \tag{16}$$

and construct the sequence

$$\bar{f}_n = (M^*)^n \bar{f}_o, \quad n = 0,1,\dots, \quad \bar{f}_o = \hat{f}_o. \tag{17}$$

By induction we find that $\bar{f}_n \leq \hat{f}_n$, $n \geq 0$. Hence for the convergence radius $R(x)$ of the series $\sum \bar{f}_n(x) z^n$ holds $\bar{R}(x) \geq$ $\geq \bar{R}(x)$. From (17) we have

$$\sum_{n=0}^{\infty} \bar{f}_n(x) z^n = \sum_{n=0}^{\infty} z^n \int m_n^* (x,y) \bar{f}_o(y) dV(y) =$$

$$= \int \left(\sum_{n=0}^{\infty} m_n^* (x,y) z^n \right) \bar{f}_o(y) dV(y).$$

But the series on the right hand side have for almost all y the radius of convergence $R(M^*)$, so we have $\bar{R}(x) \leq R(M^*)$. Finally we conclude that $R(x) \leq \bar{R}(x) \leq R(M^*) < R(x)$, which is a contradiction.

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