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FIXED POINT THEOREMS OF ROTHE-TYPE FOR FRUM-KETKOV- AND 1-SET-CONTRACTIONS

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Abstract: We prove two fixed point theorems for continuous functions $f: X \longrightarrow E$, where E is a normed linear space and X is a "nice" retract of E. Besides $f(\partial X) \subseteq X$ we assume f either to be a 1-set-contraction or to satisfy a so-called Frum-Ketkov condition. As consequences we get a theorem stated by R.L. Frum-Ketkov and results for condensing as well as for non-expansive mappings.

Key words: Fixed point theorems, Banach space, Rothe-type, Frum-Ketkov-contraction, 1-set-contraction, condensing, nonexpansive and LANE mappings.

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1. Introduction. In 1967 R.L. Frum-Ketkov [2] claimed Corollary 1 of this paper for the special case that X is the unit ball of a Banach space. But R.D. Nussbaum remarked in [6],[7] and [8] that Frum-Ketkov's proof seems to be in error. During the years a number of authors established related theorems; c.f. F.E. Browder [1, Theorem 16.3], R.D. Nussbaum [6] and [8], M. Furi and M. Martelli [3], M.A. Krasnoselskii [5] and R. Schöneberg [13], but none of these results includes Frum-Ketkov's theorem as a special case. Moreover, their proofs do not generalize to the situation of this theorem, since either they depend heavily on special

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1) This work is part of the author's thesis which has been prepared under the supervision of Prof. J. Reinermann.

space structures such as Hilbert space and π_1-space, or they use the Lefschetz number theory, hence the assumption $f(X) \subseteq X$ is necessary. So, as far as we know the first complete proof of Frum-Ketkov's theorem is given in this paper.

By using standard methods from the theory of set-contractions the proof of Theorem 1 can be modified to give the corresponding result for 1-set contractions, Theorem 2. It is followed by direct consequences for condensing maps, nonexpansive maps, and related functions.

2. Main Results. Before we state our main theorems and deduce the Corollaries, let us establish some basic notation. For a subset X of a normed linear space (abbreviated: n.l.s.) E we write ∂X, $c\ell(X)$, $co(X)$, $\overline{co}(X)$ for the boundary, closure, convex hull and closure of the convex hull of X, respectively; $d(y,X) = \inf\{\|y-x\| \mid x \in X\}$ denotes the distance of the point $y \in E$ from X; X is said to be contractible (in itself to a point) iff there is $x_0 \in X$ and $F: X \times [0,1] \to X$ continuous such that $F(x,0) = x$ and $F(x,1) = x_0$ for all $x \in X$. The collection of all nonempty subsets of E that can be written as a finite union of closed convex sets is denoted by $\mathcal{F}_0$, more precisely, $\mathcal{F}_0 = \{Y \mid \emptyset \neq Y \subseteq E$, there are $n \in \mathbb{N}$ and $C_1, \ldots, C_n \subseteq E$ closed convex such that $Y = \bigcup\{C_i \mid 1 \leq i \leq n\}\}$.

With these notations we have

Theorem 1. Let E be a n.l.s. and $X \in \mathcal{F}_0$ be contractible. Suppose $f: X \to E$ is continuous such that $f(\partial X) \subseteq X$ and

$$(1)\begin{cases}\text{there are } c \in [0,1) \text{ and } g: X \longrightarrow E \text{ (not necessarily continuous) such that } c\ell(g(X)) \text{ is compact and} \\ \| f(x)-g(x) \| \leq c \| x-g(x) \| \quad \text{for all } x \in X.\end{cases}$$

Then f has a fixed point, i.e. $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\} \neq \emptyset$.

Obviously the condition (1) of Theorem 1 is less restrictive than the corresponding condition (FK) in Corollary 1 which was introduced by R.L. Frum-Ketkov in [2] (let $g(x) \in K$ be such that $d(f(x),K) = \| f(x)-g(x) \|$ for all $x \in X$). So we get

Corollary 1. Let E be a n.l.s. and $X \in \mathcal{F}_0$ be contractible. Suppose $f: X \longrightarrow E$ is continuous such that $f(\partial X) \subseteq$ $\subseteq X$ and

$$(FK)\begin{cases}\text{there are } c \in [0,1) \text{ and } K \subseteq E \text{ compact such that} \\ d(f(x),K) \leq c\, d(x,K) \text{ for all } x \in X.\end{cases}$$

Then $\mathrm{Fix}(f) \neq \emptyset$.

The corresponding results for set-contractions need again some preparations. We shall use the Kuratowski (diameter) measure of noncompactness γ , which associates to any bounded subset A of the n.l.s. E the nonnegative real $\gamma(A) = \inf \{ d > 0 \mid$ there is a finite covering of A by sets of diameter less than or equal to $d \}$, in the definitions and proofs, but all results can be read for the Hausdorff (ball) measure of noncompactness, too. Let $c \geq 0$, a function $f: X \longrightarrow E$, where X is a subset of the n.l.s. E, is called a c-set-contraction [condensing] iff it is continuous and for each bounded subset A of X we have $f(A)$ is bounded and $\gamma(f(A)) \leq c\gamma(A)$ [$\gamma(f(A)) < \gamma(A)$ if $\gamma(A) > 0$, respect.] .

With these notations we have

Theorem 2. Let E be a n.l.s. and $X \in \mathcal{F}_0$ be contractible. Suppose $f:X \longrightarrow E$ is a 1-set-contraction such that $f(\partial X) \subseteq X$ and $f(X)$ is bounded.
Then $\inf\{\| x-f(x) \| \mid x \in X\} = 0$; consequently, if we assume $(Id-f)(X)$ to be closed, we have $Fix(f) \neq \emptyset$.

Since it is well-known (c.f. Nussbaum [10]) that if E is a Banach space, $X \subseteq E$ is closed and $f:X \longrightarrow E$ is condensing such that $f(X)$ is bounded, then $(Id-f)(X)$ is closed, we get

Corollary 2. Let E be a Banach space and $X \in \mathcal{F}_0$ be contractible. Suppose $f:X \longrightarrow E$ is condensing such that $f(\partial X) \subseteq X$ and $f(X)$ is bounded.
Then $Fix(f) \neq \emptyset$.

For nonexpansive mappings f (i.e., $\| f(x)-f(y) \| \leq \| x-y \|$ for all $x,y \in X$) being defined on closed bounded convex domains X in a uniformly convex Banach space it is known for more than 10 years (c.f. Göhde [4]) that $(Id-f)(X)$ is closed. R.D. Nussbaum introduced in [9] a more general class of mappings, the locally almost nonexpansive (abbreviated: LANE) functions: If E is a n.l.s. and $X \subseteq E$, $f:X \longrightarrow E$ is called LANE function iff for any $x \in X$ and $\varepsilon > 0$ there is a weak neighborhood N of x such that $\| f(y)-f(z) \| \leq \| y-z \| + \varepsilon$ for all $y,z \in N$. He proves in [9] that if f is a LANE function defined on a closed bounded and convex subset X of a uniformly convex Banach space E with image in E, then f is a 1-set-contraction and $(Id-f)(X)$ is closed. Obviously these results remain true for bounded $X \in \mathcal{F}_0$. Thus we have

Corollary 3. Let E be a uniformly convex Banach space and $X \in \mathcal{F}_0$ be bounded and contractible. Suppose $f: X \longrightarrow E$ is a LANE function such that $f(\partial X) \subseteq X$.

Then $\mathrm{Fix}(f) \neq \emptyset$.

For related results see [14].

3. Proofs of Theorem 1 and 2. Our proofs are based on the following Lemma, which was proved in a more general form by J. Reinermann and R. Schöneberg (c.f. [11],[12]). We will give the short proof for the sake of completeness.

Lemma 1. Let E be a n.l.s. and $X \in \mathcal{F}_0$ be contractible. Suppose $f: X \longrightarrow E$ is compact (i.e., f is continuous and $c\ell(f(X))$ is compact) such that $f(\partial X) \subseteq X$.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. X is a contractible neighborhood retract of E and hence a retract. Let $r: E \longrightarrow X$ be continuous such that $r(x) = x$ for all $x \in X$, and define $g: E \longrightarrow E$ by $g(x) = f(x)$ for $x \in X$ and $g(x) = r(f(r(x)))$ for $x \in E \setminus X$. Then g is compact and hence has a fixed point x by Schauder's fixed point theorem, since $x = g(x) \in X$ it follows $f(x) = g(x) = x$. Q.E.D.

The next Lemma is crucial for the following.

Lemma 2. Let E be a n.l.s., $\emptyset \neq X \subseteq E$, $f: X \longrightarrow E$ such that Condition (1) of Theorem 1 is fulfilled. Assume further $\inf\{\| x-f(x) \| \mid x \in X\} > 0$.

Then there are $p \in \mathbb{N}$, $(x_1, \ldots, x_p) \in E^p$ and a strictly increasing sequence (r_n) of positive reals such that $r_{n+1} - r_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and with

$\mathcal{L} = \{ \bar{B}(x_i, r_n) \mid 1 \leq i \leq p,\ n \in \mathbb{N} \}$ [1] we have

(2) for all $x \in X$ there is $C \in \mathcal{L}$ such that $x \notin C$ and $f(x) \in C$.

Remarks. 1) If X is bounded, there is obviously a finite subset $\mathcal{L}'$ of the set $\mathcal{L}$ of the conclusion of Lemma 2 such that (2) remains true if $\mathcal{L}'$ is substituted for $\mathcal{L}$.

2) R. Schöneberg [14] has defined a fixed point index for functions satisfying Condition (2) with finite $\mathcal{L}$ on the boundary of their domain. Consequences of Lemma 2 and this fixed point index will be investigated in a paper in preparation by R. Schöneberg and the author.

Proof of Lemma 2. Let $a = \inf\{ \| x - f(x) \| \mid x \in X \} > 0$ and choose $b \in (0, (1-c)a/4)$. Since $c\,\ell(g(X))$ is compact there are $p \in \mathbb{N}$ and $(x_1, \dots, x_p) \in E^p$ such that $g(X) \subseteq \subseteq \bigcup \{ \bar{B}(x_i, b) \mid 1 \leq i \leq p \}$. Let (r_n) be such that $(1-c)^{-1} 2b < < r_1 < a/2$ and $r_{n+1} = c^{-1}(r_n - (1+c)b)$ $(n \in \mathbb{N})$. Thus $r_{n+1} - - r_n = c^{-1}((1-c)r_n - (1+c)b)$ for all $n \in \mathbb{N}$, and this yields $r_{n+1} > r_n > (1+c)(1-c)^{-1} b$ for all $n \in \mathbb{N}$ by induction. Furthermore we get $r_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and even $r_{n+1} - - r_n \longrightarrow \infty$ as $n \longrightarrow \infty$. It remains to prove (2).

Let $x \in X$ and $g(x) \in \bar{B}(x_i, b)$. For $m = -1 + \min\{ n \in \mathbb{N} \mid n \geq \geq 2 \wedge x \in \bar{B}(x_i, r_n) \}$ we have: $\| f(x) - x_i \| \leq \| f(x) - g(x) \| + b \leq \leq c \| x - g(x) \| + b \leq c r_{m+1} + (1+c)b = r_m$. Hence for $C = = \bar{B}(x_i, r_m)$ we have $f(x) \in C$; $x \notin C$ follows from the definiti-

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1) For $x \in E$ and $r > 0$ let $\bar{B}(x, r) = \{ y \in E \mid \| y - x \| \leq r \}$

tion of m if $m \geq 2$ and in the case m = 1 from diameter C = $= 2r_1 < a$. Q.E.D.

The following lemma is a generalization to certain infinite systems of closed convex sets of a result which is basic for the proofs in [8] and [3].

<u>Lemma 3</u>. <u>Let</u> E be a n.l.s., m and p be positive integers, $C_1,\dots,C_m$ closed convex subsets of E, $(x_1,\dots,x_p) \in E^p$ and (r_n) a strictly increasing sequence of positive reals such that $r_{n+1} - r_n \longrightarrow \infty$ as $n \longrightarrow \infty$.
<u>Then</u> there is a map $s: E \longrightarrow E$ which is compact and finite-dimensional (i.e., $\dim(\operatorname{span} s(E)) < \infty$) satisfying the following property:

$$s(C) \subseteq C \text{ for each } C \in \mathcal{L} = \{\bar{B}(x_i,r_n) \mid 1 \leq i \leq p,\ n \in \mathbb{N}\} \cup \cup \{C_j \mid 1 \leq j \leq m\} .$$

<u>Proof</u>. Let $\mathcal{L}'$ and $\mathcal{L}_o'$ be the set of all nonempty intersections of elements of $\mathcal{L}$ and $\mathcal{L}_o = \{C_j \mid 1 \leq j \leq m\}$ respectively. For the set $\mathcal{D} = \{D \in \mathcal{L}' \mid (C \cap D \neq \emptyset \Longrightarrow D \subseteq C)$ for all $C \in \mathcal{L}\}$ we can show

(3) For each $C' \in \mathcal{L}'$ there exists $D \in \mathcal{D}$ such that $D \subseteq C'$,

(4) $\mathcal{D}$ is finite.

To prove (3) let C' be an arbitrary element of $\mathcal{L}'$ and define a sequence of sets $C_j' \subseteq E$ recursively by

$$C_o' = C' \text{ and } C_{j+1}' = \begin{cases} C_j' & \text{if } C_j' \cap C_{j+1} = \emptyset \\ C_j' \cap C_{j+1} & \text{otherwise} \end{cases} \quad \text{for } o \leq j \leq m-1$$

and

$$C_{m+i}' = C_{m+i-1}' \cap \bar{B}(x_i,r_{n_i}) \text{ where } n_i = \min\{n \in \mathbb{N} \mid C_{m+i-1}' \cap \bar{B}(x_i,r_n) \neq \emptyset\} \text{ for } 1 \leq i \leq p.$$

Then it is easy to show that C'_{m+p} is an element of $\mathcal{D}$ which is contained in C'.

To prove (4) select $y_{C'} \in C'$ for each $C' \in \mathcal{L}'_o$ and let $k \in \mathbb{N}$ be such that $y_{C'} \in \bar{B}(x_i, r_k)$ and $r_k - r_{k-1} \geq \| x_i - x_j \|$ for $1 \leq i,\ j \leq p$ and $C' \in \mathcal{L}'_o$. Then we have

(5) $\bar{B}(x_i, r_k) \supseteq \bar{B}(x_j, r_n)$ for all $1 \leq i,\ j \leq p$ and $n < k$.

We show $\mathcal{D} \subseteq \{ C'_o \cap \bigcap_{i=1}^{n} \bar{B}(x_i, r_{n_i}) \mid C'_o \in \mathcal{L}'_o,\ (n_1, \ldots, n_p) \in \{1, \ldots, k\}^p \}$ which obviously implies (4). Let $D \in \mathcal{D}$ and define $C'_o = \bigcap \{ C \in \mathcal{L}_o \mid D \subseteq C \}$ and $n_i = \min \{ n \in \mathbb{N} \mid D \subseteq \bar{B}(x_i, r_n) \}$ $(1 \leq i \leq p)$. Since $D \in \mathcal{L}'$ it follows $D = C'_o \cap \bigcap_{i=1}^{n} \bar{B}(x_i, r_{n_i})$, so it remains to show that $n_i \leq k$ for $1 \leq i \leq p$.

Otherwise there is some $i \in \{1, \ldots, p\}$ such that $n_i > k$. This, together with (5) and the definition of n_i, implies $n_j \geq k$ for all $j \in \{1, \ldots, p\}$. But then $y_{C'_o} \in C'_o \cap \bigcap_{j=1}^{n} \bar{B}(x_j, r_k) \subseteq D \cap \bar{B}(x_i, r_k)$ and, by the definition of $\mathcal{D}$, $D \subseteq \bar{B}(x_i, r_k)$, a contradiction to $n_i > k$. Thus (4) is proved.

Since for each $D \in \mathcal{D}$ the set $\{ C \in \mathcal{L} \mid C \cap D = \emptyset \}$ is finite, the set $U_D = E \setminus \bigcup \{ C \in \mathcal{L} \mid C \cap D = \emptyset \}$ is an open neighborhood of D. We show $\bigcup \{ U_D \mid D \in \mathcal{D} \} = E$. Let $x \in E$ and define $C' = \bigcap \{ C \in \mathcal{L} \mid x \in C \}$. Since $x \in C'$ and hence $C' \in \mathcal{L}'$, by (3) there is $D \in \mathcal{D}$ such that $D \subseteq C'$. So we have $(x \in C \Longrightarrow D \subseteq C)$ for all $C \in \mathcal{L}$ and hence $x \in U_D$.

Let $(h_D)_{D \in \mathcal{D}}$ be a partition of unity subordinate to the finite cover $(U_D)_{D \in \mathcal{D}}$ of E and select $z_D \in D$ for each $D \in \mathcal{D}$. Define $s : E \longrightarrow E$ by $s(x) = \sum_{D \in \mathcal{D}} h_D(x) z_D$. Obviously,

$s(E) \subseteq \mathrm{co}(\{ z_D \mid D \in \mathcal{D} \})$ and hence s is compact and finite-dimensional. Now let $C \in \mathcal{L}$ and $x \in C$. Then for all $D \in \mathcal{D}$: $h_D(x) > 0$ implies $x \in U_D$ and hence $C \cap D \neq \emptyset$ and, by definition of $\mathcal{D}$, $D \subseteq C$. Hence we have $s(x) \in \mathrm{co}(\{ z_D \mid D \in \mathcal{D}, D \subseteq C \}) \subseteq C$ and our proof is complete. Q.E.D.

Remark. If one only needs $s: E \longrightarrow E$ compact, finite-dimensional such that $s(C) \subseteq C$ for all $C \in \mathcal{L}$, where $\mathcal{L}$ is a finite system of closed convex subsets of E, the proof is considerably less complicated since the properties (3) and (4) of $\mathcal{D}$ are obvious in this case. The remaining proof is essentially the same as in [3].

After these preparations we can prove our first main result.

Proof of Theorem 1. We shall first show that $a = \inf \{ \| x-f(x) \| \mid x \in X \} = 0$. The proof of this is by contradiction. Assume $a > 0$. By Lemma 2 and 3 there is a system $\mathcal{L}$ of closed convex subsets of E and a compact map $s: E \longrightarrow E$ such that for each $x \in X$ there is $C \in \mathcal{L}$ such that $f(x) \in C$ and $x \notin C$, and $s(C) \subseteq C$ for all $C \in \mathcal{L}$ as well as $s(X) \subseteq X$ (recall $X \in \mathcal{F}_0$). The function $\bar{g}: X \longrightarrow E$ defined by $g(x) = s(f(x))$ is obviously compact and satisfies $\bar{g}(\partial X) \subseteq s(X) \subseteq X$. By Lemma 1 there is $x \in X$ such that $x = \bar{g}(x) = s(f(x))$. Select $C \in \mathcal{L}$ such that $x \notin C$ and $f(x) \in C$, then $x = s(f(x)) \in s(C) \subseteq C$, a contradiction. Thus we have $a = 0$.

By hypothesis (1) we have for all $x \in X$, $\| x-g(x)) \| \leq \| x-f(x) \| + c \| x-g(x) \|$ and thus $\| x-g(x) \| = (1-c)^{-1} \| x-f(x) \|$. Let (x_n) be a sequence in X and $y \in E$ such that $\| x_n-f(x_n) \| \longrightarrow 0$ and $g(x_n) \longrightarrow y$ as $n \longrightarrow \infty$. Hence we get $x_n \longrightarrow y$ and $f(x_n) \longrightarrow y$ as $n \longrightarrow \infty$ and, since X is closed and f is continuous, $f(y) = y$. Q.E.D.

As a preparation for the proof of Theorem 2 let us establish the following Lemma, which corresponds in some sense to Lemma 2.

Lemma 4. Let E be a n.l.s. and $\emptyset \neq X \subseteq E$. Suppose $f: X \longrightarrow E$ is a c-set-contraction where $c \in (0,1)$ such that $f(X)$ is bounded. Assume $\inf \{ \| x-f(x) \| \mid x \in X \} > 0$. Then there is a finite system $\mathcal{L}$ of closed convex subsets of E such that for all $x \in X$ there is $C \in \mathcal{L}$ such that $f(x) \in C$ and $x \notin C$.

Remark. If X is closed and E is complete the hypothesis $\inf \{ \| x-f(x) \| \mid x \in X \} > 0$ is equivalent to $Fix(f) = \emptyset$ and this Lemma shows that R. Schöneberg's fixed point index [14] (c.f. Remark to Lemma 2) is applicable to c-set-contractions ($0 \leq c < 1$) which have no fixed points on the boundary of their domain. Lemma 2 and the last part of the proof of Theorem 1 show the corresponding for Frum-Ketkov-contractions on bounded domains.

Proof of Lemma 4. Let $C_1 = \overline{co}(f(X))$ and $C_{n+1} = \overline{co}(f(X \cap C_n))$ for $n \in \mathbb{N}$. Since $\gamma(C_{n+1}) \leq c\, \gamma(C_n) \leq c^n \gamma(C_1)$ for all $n \in \mathbb{N}$, we can choose $k \in \mathbb{N}$ such that $\gamma(C_{k+1}) < a = \inf \{ \| x-f(x) \| \mid x \in X \}$. Let $\mathcal{D}$ be a finite covering of C_{k+1} by closed convex subsets of E of diameter less than a. Let $x \in X$. If $x \in C_k$ there is $D \in \mathcal{D}$ such that $f(x) \in D$; since the diameter of D is less than a we have $x \notin D$. If $x \notin C_k$ let $m = \min \{ n \in \mathbb{N} \mid 1 \leq n \leq k,\ x \notin C_n \}$, then

$f(x)\in C_m$ and $x\notin C_m$. Hence $\mathcal{L} = \mathcal{D} \cup \{C_n \mid 1\le n\le k\}$ is the required system. Q.E.D.

Proof of Theorem 2. Suppose $a = \inf\{\|x-f(x)\| \mid x\in X\} > 0$. Let $m\in \mathbb{N}$ and $C_1,\dots,C_m$ be closed convex subsets of E such that $X = \bigcup\{C_i \mid 1\le i\le m\}$. By Lemma 3 there is $\tilde{s}:X\to X$ compact, such that $\tilde{s}(C_i)\subseteq C_i$ for all $i\in\{1,\dots,m\}$. Since $f(X)$ is bounded there is $c\in(0,1)$ such that for all $x\in X$ we have $\|f(x) - (cf(x) + (1-c)\tilde{s}(f(x)))\| = (1-c)\|f(x) - \tilde{s}(f(x))\|\le a/2$. Let $g:X\to E$ be defined by $g(x) = cf(x) + (1-c)\tilde{s}(f(x))$. Then we have $\|x-g(x)\|\ge \|x-f(x)\| - \|f(x) - g(x)\|\ge a/2$ for all $x\in X$. Furthermore, for each bounded subset A of X we have $g(A)\subseteq cf(A) + (1-c)\tilde{s}(f(A))$ and, since $\tilde{s}$ is compact and f is 1-set-contraction, $\gamma(g(A))\le c\gamma(f(A))\le c\gamma(A)$. Thus g is a c-set-contraction. For any $x\in\partial X$ there is $i\in\{1,\dots,m\}$ such that $f(x)\in C_i$ and hence $\tilde{s}(f(x))\in\tilde{s}(C_i)\subseteq C_i$. Thus $g(\partial X)\subseteq X$.

By Lemma 3 and Lemma 4 there is a finite system $\mathcal{L}$ of closed convex subsets of E and a compact map $s:E\to E$ such that for all $x\in X$ there is $C\in\mathcal{L}$ such that $g(x)\in C$ and $x\notin C$ and $s(C')\subseteq C'$ for all $C'\in\mathcal{L}\cup\{C_i \mid 1\le i\le m\}$. The map $h:X\to E$ defined by $h(x) = s(g(x))$ satisfies all the hypothesis of Lemma 1 and hence there is $x\in X$ such that $x = s(g(x))$. But there is $C\in\mathcal{L}$ such that $g(x)\in C$ and $x\notin C$, a contradiction to $s(C)\subseteq C$. Thus $a = 0$, i.e., $0\in c\ell((Id - f)(X))$; hence, if we assume $(Id - f)(X)$ to be closed, we have $0\in(Id - f)(X)$, i.e., $Fix(f)\neq\emptyset$. Q.E.D.

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