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# SEPARATION OF TWO CONVEX SETS BY OPERATORS

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Abstract: In this paper we generalize the usual separation theorems (where the separation is carried out by (continuous) linear functionals) to separation involving (continuous) linear operators mapping a (topological) vector space in a (normal topological) partially ordered vector space.

Key words: Separation theorems, convex sets.

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§ 1. Introduction. In our paper [8] some assertions on separation of convex sets by means of linear operators are published, where a suitable formulation of the Hahn-Banach theorem for operators is used (cf. also [13] and [14]).

In this paper we develop a different approach for the proofs of such separation theorems using the separation of convex sets by linear functionals.
Simultaneously, the theorems of this note contain classical assertions as special cases.

The proof principle given here is more convenient than that in [8], since this new way of the proofs is easily transformable to continuous linear operators. Moreover, we can prove stronger assertions than in [8].

§ 2. <u>Notions and terminology</u>. The used terminology corresponds to that in [1],[5],[13] and [16]. All vector spaces considered in the following are real vector spaces.

We will use the notations "partially ordered vector space" and "topological (partially ordered) vector space" in the common sense (see [13]).

Moreover we apply the abbreviations:

$x \lneq y$ iff $x \leqq y$ and $x \neq y$,

$F_+ := \{ x \mid x \in F,\ x \geqq 0 \}$,

$F_{++} := F_+ \setminus \{ 0 \}$.

The set $F_+$ is a proper convex cone, that means

$$F_+ \cap (-F_+) = \{0\} \ ; \ F_+ + F_+ = F_+ \ .$$

A (partially ordered) vector space F is said to be topological, if a Hausdorff-topology is defined in F such that the mappings

$$(x,y) \longrightarrow x + y \text{ of } F \times F \text{ in } F$$

and

$$(\lambda, x) \longrightarrow \lambda x \text{ of } R \times F \text{ in } F$$

are continuous (and $F_+$ is closed in this topology).

We say that a (topological) partially ordered vector space F has the <u>least upper bound property</u>, if each non-empty subset of F possessing an upper bound in F also possesses a least upper bound in F.

A topological partially ordered vector space F is said to be <u>normal</u>, if there exists a basis of neighbourhoods of 0 such that each element V of that basis has a representation in the following way:

$$V = (V + F_+) \cap (V - F_+).$$

Let E be a real vector space. For a non-empty subset A of E denote

${}^{l}A$ the affine manifold spanned by A,

${}^{i}A$ the intrinsic core of A:

$${}^{i}A := \{x \in E \mid \forall y \in {}^{l}A \, \exists \varepsilon := \varepsilon(x,y) > 0 : x + r(y - x) \in A \ \forall r \in [-\varepsilon, \varepsilon]\}$$

(if ${}^{l}A = E$ we will write $A^{i} := {}^{i}A$ for the core of A),

${}^{a}A$ the set of points in E linear attainable from A:

$${}^{a}A := \{x \in E \mid \exists y \in A, y \neq x : y + t(x - y) \in A \ \forall t \in [0,1)\}.$$

Furthermore, we define

$${}^{b}A := A \cup {}^{a}A, \quad {}^{n}A := {}^{b}A \setminus {}^{i}A.$$

For a non-empty subset A of a topological vector space E denotes int A the set of all interior points of A. Let both E and F be (topological) vector spaces. Then $\mathcal{L}(E,F)$ and $\mathcal{L}'(E,F)$ denote the real vector spaces of all linear operators and continuous linear operators $L: E \longrightarrow F$, respectively. In particular, we will write

$$E^* := \mathcal{L}(E,R), \quad E' := \mathcal{L}'(E,R)$$

for the algebraical dual space and (topological) dual space of E, respectively, where the vector space R of the reals has the usual topology. $\langle u,x \rangle$ denotes the value of $u \in E^*$ or $u \in E'$ in the point $x \in E$.

§ 3. Separation Theorems. First of all we will give the basic idea of the proofs of the following separation theorems.

Let E be a (topological) vector space, and let F be a (normal topological) partially ordered vector space, let $y_0 \in F_+$ and $y_0 \in F_{++}$, respectively. If $u \in E^*$ ($u \in E'$), $u \neq 0$, separates the sets A and B in a certain sense, then the operator $L \in \mathcal{L}(E,F)$ ($L \in \mathcal{L}'(E,F)$) defined by

$$(1) \qquad L(x) := \langle u,x \rangle \, y_0, \quad x \in E,$$

is a linear (continuous) operator which separates these sets in a certain sense. The continuity of L follows from the continuity of $u \in E'$, the compatibility of the topology in F with the vector structure and the normality of this topology.

Now let E and F be vector spaces as given above. Then from the linearity of $L \in \mathcal{L}(E,F)$ and $L \in \mathcal{L}'(E,F)$, respectively, it is easy to see that we have the following implications for two non-empty subsets A and B of E.

1. $\left.\begin{array}{l} y_0 \in F \\ L(x) \leqq y_0 \leqq L(y) \;\; \forall (x,y) \in A \times B \end{array}\right\} \Longrightarrow \; 0 \leqq L(z) \;\; \forall z \in B - A.$

2. $\left.\begin{array}{l} y_0 \in F \\ L(x) \leq y_0 \leq L(y) \;\; \forall (x,y) \in A \times B \\ \text{or} \\ L(x) \leq y_0 \leqq L(y) \;\; \forall (x,y) \in A \times B \end{array}\right\} \Longrightarrow \; 0 \leq L(z) \;\; \forall z \in B - A.$

3. $\left.\begin{array}{l} y_0 \in F \\ L(x) \leqq y_0 \leqq L(y) \ \forall (x,y) \in A \times B \\ \text{and} \\ \exists\, x \in A:\ L(x) \lneqq y_0 \\ \text{or} \\ \exists\, y \in B:\ y_0 \lneqq L(y) \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} 0 \leqq L(z)\ \forall z \in A \times B, \\ \exists\, z \in B - A:\ 0 \lneqq L(z). \end{array}\right.$

These implications hold true if a $y_0 \in F$ with the properties assumed in the implications doesn´t exist, but $L(x)$ and $L(y)$ for all $(x,y) \in A \times B$ are comparable as demanded above.

The conversions of the implications hold true for $F = R$ (see [9]). However, this is not true for operators $L \in \mathscr{L}(E,F)$ and $L \in \mathscr{L}'(E,F)$, respectively, except that $F$ has additional properties (e.g., if $F$ has the least upper bound property). Therefore, if two sets $A$ and $B$ of $E$ are separated by a linear (continuous) operator in a certain sense, then an analogous separation of $0$ and $A - B$ is possible, but not necessarily conversely.

3.1. <u>Separation by Linear Operators</u>. In this section let $E$ be a vector space and let $F$ be a partially ordered vector space with $F_{++} \neq \emptyset$.
We obtain as a stronger result than in [8], Folgerung 10,

<u>Theorem 1</u>: Let $A$ and $B$ be two non-empty convex subsets of $E$ and $B^i \neq \emptyset$. There exist an $L \in \mathscr{L}(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ with

(2) $$L(x) \leqq y_1 \leqq L(y) \quad \forall (x,y) \in A \times B,$$

(3) $$y_1 \leqq L(y) \quad \forall y \in B^i$$

if and only if $A \cap B^i = \emptyset$.

Proof: 1. Let $A \cap B^i = \emptyset$. Then there exist a $u \in E^*$, $u \neq 0$, and an $\alpha \in R$ such that

$$\langle u,x \rangle \leqq \alpha \leqq \langle u,y \rangle \quad \forall (x,y) \in A \times B,$$
$$\alpha < \langle u,y \rangle \quad \forall y \in B^i$$

(see [10], § 17, or [11]). Hence we obtain

$$\langle u,x \rangle \; y_0 \leqq \alpha y_0 \leqq \langle u,y \rangle \; y_0 \quad \forall (x,y) \in A \times B$$
$$\alpha y_0 \leqq \langle u,y \rangle \; y_0 \quad \forall y \in B^i$$

for $y_0 \in F_{++}$. $L \in \mathcal{L}(E,F)$ defined by (1) and $y_1 := \alpha y_0$ are convenient.

2. If there exists an operator $L \in \mathcal{L}(E,F)$ which has the properties (2) and (3), then the assumption $A \cap B^i \neq \emptyset$ leads to a contradiction.

Theorem 2: Let A and B be two non-empty subsets of E and $A = A^i$, $B = B^i$. There exist an $L \in \mathcal{L}(E,F)$ and a $y_1 \in F_+ \cup$ $\cup (-F_+)$ with

$$L(x) \leqq y_1 \leqq L(y) \quad \forall (x,y) \in A \times B$$

if and only if $A \cap B = \emptyset$.

Proof: Let $A \cap B = \emptyset$. Then, by [10], there exist a $u \in E^*$, $u \neq 0$, and an $\alpha \in R$ such that

$$\langle u,x \rangle < \alpha < \langle u,y \rangle \quad \forall (x,y) \in A \times B.$$

For $y_0 \in F_{++}$ it follows

$$\langle u,x \rangle \; y_0 \leqq \alpha y_0 \leqq \langle u,y \rangle \; y_0 \quad \forall (x,y) \in A \times B.$$

Therefore, $L \in \mathcal{L}(E,F)$ defined by (1) and $y_1 := \alpha y_o$ are convenient. The proof of the conversion is trivial.

In the next separation theorems we need only assumptions about the intrinsic core of certain sets. Therefore, we prove two assertions about such sets.

<u>Lemma 1</u>: Let $A \subseteq E$ and ${}^iA \neq \emptyset$, let $L \in \mathcal{L}(E,F)$ and $y_o \in F$. If

$$(4) \qquad L(x) \leqq y_o \quad \forall x \in A,$$

then

$$\exists x \in A: L(x) \leq y_o \Longleftrightarrow \{x \in E \mid L(x) = y_o\} \cap {}^iA = \emptyset.$$

<u>Proof</u>: Let $x_1 \subseteq A$ and $L(x_1) \leq y_o$. We assume that there exists an

$$x_o \in \{x \in E \mid L(x) = y_o\} \cap {}^iA.$$

Because of $x_o \in {}^iA$, for $x \in {}^1A$ there exists an $\varepsilon > 0$ such that

$$x_o + r(x - x_o) \in A \qquad \forall r \in [-\varepsilon, \varepsilon] .$$

This holds true in particular for $x = x_1$, $x_1 \in A \setminus \{x \in E \mid L(x) = y_o\}$.

Therefore, from (4) we obtain

$$L(x_o + r(x_1 - x_o)) = L(x_o) + r[L(x_1) - L(x_o)] \leqq y_o$$
$$\forall r \in [-\varepsilon, \varepsilon] .$$

Since the relation $\leqq$ is antisymmetric and $L(x_o) = y_o$, we have

$$L(x_1) = L(x_o) = y_o.$$

This is a contradiction.

The proof of the conversion of the statement is trivial.

<u>Lemma 2</u>: Let $A \subseteq E$ and ${}^{i}A \neq \emptyset$. If there exist a $u \in E^{*}$, $u \neq 0$, and an $\alpha \in R$ such that

$$\langle u,x \rangle \lneq \alpha \qquad \forall x \in A,$$

then for the operator L defined by (1) and $y_1 := \alpha y_0$, $y_0 \in F_{++}$, holds true

$$\exists x \in A: \langle u,x \rangle < \alpha \iff L(x) \leq y_1 \qquad \forall x \in {}^{i}A.$$

<u>Proof</u>: Let $x_1 \in A$ and $\langle u,x_1 \rangle < \alpha$ . Then for $x_1 \in A \subseteq {}^{1}A$ there exists an $\varepsilon > 0$ such that

$$x_o + r(x_1 - x_o) \in A \qquad \forall r \in [-\varepsilon, \varepsilon] .$$

Hence,

$$\langle u,x_o \rangle + r \langle u,x_1 - x_o \rangle = \alpha + r [\langle u,x_1 \rangle - \langle u,x_o \rangle] \leqq \alpha$$

for all r of $[-\varepsilon, \varepsilon]$. It follows

$$\langle u,x_1 \rangle = \langle u,x_o \rangle = \alpha ;$$

but this is a contradiction and, therefore, we have

$$\langle u,x \rangle < \alpha \quad \forall x \in {}^{i}A.$$

From (1) and $y_1 := \alpha y_o$, $y_o \in F_{++}$ our statement follows. It is easy to see that the conversion holds true.

In connection with these assertions we obtain theorems which are sharper than those in [8].

<u>Theorem 3</u>: Let A and B be two non-empty convex subsets of E and ${}^{i}B \neq \emptyset$. When the deficiency of B is finite, there exist an $L \in \mathcal{L}(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ with

(5) $$L(x) \leqq y_1 \leqq L(y) \qquad \forall (x,y) \in A \times B$$

and

(6) $L(x) \leq y_1$ for at least one $x \in A$

or

(7) $y_1 \leq L(y) \quad \forall y \in {}^iB$

if and only if $A \cap {}^iB = \emptyset$.

<u>Proof</u>: 1. Let $A \cap {}^iB = \emptyset$. Then there exist a $u \in E^*$, $u \neq 0$, and an $\alpha \in R$ such that

$$\langle u,x \rangle \leqq \alpha \leqq \langle u,y \rangle \quad \forall (x,y) \in A \times B$$

and

$$\langle u,x \rangle < \alpha \text{ for at least one } x \in A$$

or

$$\langle u,y \rangle > \alpha \text{ for at least one } x \in B$$

(see [3],[11]).

For $y_0 \in F_{++}$ we get

$$\langle u,x \rangle y_0 \leqq \alpha y_0 \leqq \langle u,y \rangle y_0 \quad \forall (x,y) \in A \times B$$

and

$$\langle u,x \rangle y_0 \leq \alpha y_0 \text{ for at least one } x \in A$$

or

$$\langle u,y \rangle y_0 \geq \alpha y_0 \text{ for at least one } y \in B.$$

Therefore, by Lemma 2, $L \in \mathcal{L}(E,F)$ defined by (1) and $y_1 :=$ $:= \alpha y_0$ are convenient.

2. The assumption $A \cap {}^iB \neq \emptyset$ in connection with (5),(6) and (7) leads to a contradiction.

As a generalization of Satz 4 in [8] we have

Theorem 4: Let A and B be two convex subsets of E with ${}^{i}A \neq \emptyset$ and ${}^{i}B \neq \emptyset$. There exist an $L \in \mathcal{L}(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ with

$$L(x) \leqq y_1 \leqq L(y) \qquad \forall (x,y) \in A \times B$$

and

$$L(x) \leq y_1 \qquad \forall x \in {}^{i}A$$

or

$$L(y) \geq y_1 \qquad \forall x \in {}^{i}B$$

if and only if ${}^{i}A \cap {}^{i}B = \emptyset$.

The proof of this theorem is analogous to that of Theorem 3 (with respect to [3],[9] and [11]).

As a general geometric version of the Hahn-Banach theorem we obtain

Theorem 5: Let A be a convex subset of E with ${}^{i}A \neq \emptyset$, and let M be a linear manifold in E.
There exist an $L \in \mathcal{L}(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ such that

$$L(x) \leqq y_1 = L(y) \qquad \forall (x,y) \in A \times M,$$
$$L(x) \leq y_1 \qquad \forall x \in {}^{i}A$$

if and only if $M \cap {}^{i}A = \emptyset$.

Proof: Since $M = {}^{l}M = {}^{i}M$, from [1] and our assumptions it follows that there exist a $u \in E^*$, $u \neq 0$, and an $\alpha \in R$ such that

$$\langle u,x \rangle \leqq \alpha = \langle u,y \rangle \qquad \forall (x,y) \in A \times M,$$
$$\langle u,x \rangle < \alpha \qquad \forall x \in {}^{i}A.$$

From this we get in connection with $y_0 \in F_{++}$ our statement in the same way as in the other proofs.

The assertion of this theorem is also proved in [8], Folgerung 11, however, different methods have been used. Moreover, here the assumption that F is archimedean, can be dropped. In [2] a new proof of this statement is also given.

We have an analogous result for $M \subseteq {}^{n}A$ in

Theorem 6: Let $A \subseteq E$ and $M \subseteq {}^{n}A$ convex sets with ${}^{i}A \neq \emptyset$ and ${}^{i}M \neq \emptyset$. Then there exist an $L \in \mathcal{L}(E,F)$ and a $y_1 \in F_+ \cup \cup (-F_+)$ such that

$$L(x) \leqq y_1 = L(y) \qquad \forall (x,y) \in A \times M,$$
$$L(x) \lneq y_1 \qquad \forall x \in {}^{i}A.$$

For the proof of this statement, a theorem given in [1] may be used.

A sharper result than in [8], Folgerung 9, is contained in the following theorem in which the strong separation is generalized.

Theorem 7: Let A be a convex subset of E with ${}^{i}A \neq \emptyset$, and let $x_0 \in E \setminus {}^{b}A$. Then there exist an $L \in \mathcal{L}(E,F)$ and $y_1, y_2 \in F_+$ or $y_1, y_2 \in -F_+$ such that

$$L(x) \leqq y_1 \leq y_2 \leqq L(x_0) \qquad \forall x \in A.$$

Proof: By our assumptions there exist a $u \in E^*$, $u \neq 0$, an $\alpha \in R$ and an $\varepsilon > 0$ such that

$$\langle u,x \rangle \leqq \alpha - \varepsilon < \alpha \leqq \langle u,x_0 \rangle \qquad \forall x \in A$$

(cf. [1]). For $y_0 \in F_{++}$ it follows

$$\langle u,x \rangle y_0 \leqq (\alpha - \varepsilon) y_0 \lneq \alpha y_0 \leqq \langle u,x_0 \rangle y_0 \qquad \forall x \in A.$$

Therefore $y_1 := (\alpha - \varepsilon) y_0$, $y_2 := \alpha y_0$ and $L \in \mathcal{L}(E,F)$ defined by (1) are convenient.

3.2. Separation by Means of Continuous Linear Operators. For some theorems of the last section, analogous theorems with continuous operators separating certain sets, can be proved. In this section let E be a topological vector space and let F be a normal topological partially ordered vector space with $F_{++} \neq \emptyset$.

In the beginning of Section 3 it was said in which way the continuity of an operator defined in (1) follows. Therefore, the proofs can be finished if there exists a continuous linear functional $u \in E'$, $u \neq 0$, which has certain separation properties.

As a generalization of a theorem proved in [14] by means of other methods we have

Theorem 1': Let A and B be two non-empty convex subsets of E and int $B \neq \emptyset$. There exist an $L \in \mathcal{L}'(E,F)$ and a $y_1 \in F_+ \cup \cup (-F_+)$ with

$$L(x) \not\geq y_1 \not\geq L(y) \qquad \forall (x,y) \in A \times B,$$
$$y_1 \not\leq L(y) \qquad \forall y \in \text{int } B$$

if and only if $A \cap \text{int } B = \emptyset$.

The proof goes in the same way as for Theorem 1. The existence of a $u \in E'$, $u \neq 0$, and an $\alpha \in R$ having the demanded properties follow from [10].

Analogously to Theorem 2 we get

Theorem 2': Let A and B be two non-empty convex subsets of E with int A = A and int B = B.

There exist an $L \in \mathcal{L}'(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ with

$$L(x) \not\leq y_1 \not\leq L(y) \qquad \forall (x,y) \in A \times B$$

if and only if $A \cap B = \emptyset$.

The proof, using a result from [10], is trivial.

As a general geometric version of the Hahn-Banach theorem including an assertion for a continuous linear operator we have

<u>Theorem 5'</u>: Let A be a convex subset of E with int $A \neq \emptyset$, and let M be a linear manifold in E.
There exist an $L \in \mathcal{L}'(E,F)$ and a $y_1 \in F_+ \cup (-F_+)$ such that

$$L(x) \leqq y_1 = L(y) \qquad \forall (x,y) \in A \times M,$$
$$L(x) \lneq y_1 \qquad x \in \text{int } A$$

if and only if $M \cap \text{int } A = \emptyset$.

A result from [10] can be used for the proof.
Then it is easy to see

$y_1 := \alpha y_0$ with $y_0 \in F_{++}$ and $L \in \mathcal{L}'(E,F)$ defined in (1)

are convenient.

If E is a locally convex topological vector space, then strong separation theorems for compact subsets can be proved, too.

<u>Theorem 8</u>: Let A and B be closed convex subsets of a locally convex topological vector space E, and let A be compact. Then there exist an $L \in \mathcal{L}'(E,F)$ and $y_1, y_2 \in F_+$ or $y_1, y_2 \in -F_+$ such that

$$L(x) \leqq y_1 \lneq y_2 \leqq L(y) \qquad \forall (x,y) \in A \times B.$$

Proof: By our assumptions there exist a $u \in E'$, $u \neq 0$, an $\alpha \in R$ and an $\varepsilon > 0$ such that

$$\langle u,x \rangle \leqq \alpha - \varepsilon < \alpha \leqq \langle u,y \rangle \qquad \forall (x,y) \in A \times B$$

(cf. [6], section V.2). For $y_o \in F_{++}$ in connection with (1), $y_1 := (\alpha - \varepsilon) y_o$ and $y_2 := \alpha y_o$ the statement follows.

As corollaries from Theorem 8 we find the following assertions.

1. If $A \subset E$ is a closed convex set and $x_o \notin A$, then there exist an $L \in \mathscr{L}'(E,F)$ and $y_1, y_2 \in F_+$ or $y_1, y_2 \in - F_+$ such that

$$L(x) \leqq y_1 \leq y_2 = L(x_o) \qquad \forall x \in A.$$

2. For any $x_1, x_2 \in E$, $x_1 \neq x_2$, there exists an $L \in \mathscr{L}'(E,F)$ such that

$$L(x_1) \leqq L(x_2) \text{ and } L(x_1) \neq L(x_2).$$

Remarks:

1. The assumptions for Theorem 8 can be weakened e.g. in the following way:

Let A and B be closed convex subsets of a locally convex topological vector space E, let A be a continuous subset and one of the sets A, B locally compact.

Then the assertion of Theorem 8 holds true.

For a proof of this statement, a result given in [9], Theorem 2.9, can be used.

2. Using in (1) $y_o \in F_+^i$ it is possible to derive further separation theorems for operators in accordance with the usual strong separation.

For instance, if F is finite-dimensional and $F_+^i \neq \emptyset$, then Theorem 7 holds true in the following sharper form:

Let A be a convex subset of E with ${}^iA \neq \emptyset$ and let $x_o \in E \setminus {}^bA$. Then there exist an $L \in \mathscr{L}(E,F)$ and $y_1, y_2 \in F_+$ or

$y_1, y_2 \in -F_+$ such that $L(x) \leqq y_1 < y_2 \leqq L(x_0) \quad \forall x \in A$.
To this $y_1 < y_2$ is defined by $y_2 - y_1 \in F_+^i$.

3. It is a consequence from our proof principle that every classical separation theorem leads to an analogous separation theorem for linear operators.

4. It is easy to see that some separation theorems proved in this paper can be generalized on finite families of convex sets (cf. [15]).

5. Analytic versions of the Hahn-Banach theorem and some equivalent assertions were considered in [12].

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