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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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THE HAHN-BANACH PROPERTY AND EQUIVALENT CONDITIONS

Reinhard NEHSE, Halle/Saale

Abstract: Several general properties are proved to be equivalent to Hahn-Banach extension property in a partially ordered vector space. The properties include the least upper bound property, the separation property and modified Farkas-Minkowski or Kuhn-Tucker or Krein properties.

Key words: Partially ordered vector space, Hahn-Banach theorem, separation property.

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§ 1. Introduction. The Hahn-Banach theorem is known to have fundamental importance for several fields in mathematics, for instance in functional analysis, convex analysis and mathematical optimization. Further, it is a well-known fact (see To [9]) that the least upper bound property (lub) of a real partially ordered vector space F (this means that every nonempty subset of F which has in F an upper bound, has also in F a least upper bound) is equivalent to the Hahn-Banach extension property (HB): for a sublinear mapping $T:E \longrightarrow F$ and a linear mapping $L_0:A \longrightarrow F$ with $L_0(x) \leqq T(x)$ for all $x \in A$, where A is a subspace of the real vector space E, there exists a linear mapping $L:E \longrightarrow F$ such that $L_0(x) = L(x)$ for all $x \in A$ and $L(x) \leqq T(x)$ for all $x \in E$.

Previously Day [2] and Elster/Nehse [3],[4] have discussed some conditions which are equivalent to (lub).

The purpose of this paper is to prove more general equivalent conditions. By this we are able to give applications to nonconvex analysis. Our general separation theorem for sets in a product space leads to generalizations of several well-known theorems.

§ 2. Notations and terminology. Throughout this paper R denotes the field of real numbers ordered in the usual sense, E denotes a real vector space and F denotes a real partially ordered vector space, that is a vector space, where a binary reflexive, transitive and antisymmetrical relation " $\leqq$ " is defined which is compatible with the vector structure of F. E(K) denotes a real vector space quasiordered by the convex cone K with $0 \in K$ as a vertex.

Further, we apply some abbreviations: $F_+ := \{y \in F / 0 \leqq y\}$; $\mathcal{L}(E,F)$ denotes the real vector space of all linear operators $L: E \longrightarrow F$;

$$\mathcal{L}_+(E(K),F) := \{L \in \mathcal{L}(E(K),F) / 0 \leqq L(y) \quad \forall y \in K\}.$$

Now let C be a nonempty subset of a real vector space. Then ${}^{l}C$ denotes the affine manifold spanned by C; ${}^{i}C$ denotes the algebraical relative interior of C, that is

$${}^{i}C := \{u \in C / \forall v \in {}^{l}C \ \exists \ t \in R_+, \ t \neq 0: \ u + r(v-u) \in C \ \forall r \in (-t,t)\}.$$

C is said to be expansive if for at least one $u_0 \in {}^{i}C$ and every $u \in C$ holds $u_0 + t(u - u_0) \in {}^{i}C$ for all $t \in [0,1)$. For a mapping $T: C \longrightarrow F$ we define

$$\text{epi } T := \{(u,z) \in C \times F / T(u) \leqq z\},$$

$$\text{hypo } T := \{(u,z) \in C \times F / z \leqq T(u)\}.$$

Moreover, we use the following notations for a non-empty subset C of $E \times F$:

$$C(C) := \{ z \in E \times F / z = tu,\ t \in R_+,\ u \in C \}$$

as the cone spanned by C;

$$P_E(C) := \{ x \in E / \exists\ y \in F: (x,y) \in C \}$$

as the E-projection of C, where P_E is a mapping defined by $P_E(x,y) = x$ for all $(x,y) \in E \times F$.

§ 3. A separation theorem. We will say that F has the separation property (S), if in F holds true:

Let A and B be subsets of $E \times F$ such that $C(A - B)$ is convex, $P_E(A - B)$ is expansive [1)] and

(1) $0 \in {}^i P_E(A - B)$.

Then there exist an $L \in \mathcal{L}(E,F)$ and a $y_0 \in F$ such that

(2) $$L(x_1) - y_1 \not\geq y_0 \not\geq L(x_2) - y_2 \quad \forall (x_1,y_1) \in A,\ \forall (x_2,y_2) \in B$$

if and only if

(3) $$\left.\begin{array}{l}(x,y_1) \in A \\ (x,y_2) \in B\end{array}\right\} \Longrightarrow y_2 \not\geq y_1.$$

Theorem 1. If F has the least upper bound property, then F has the separation property.

Proof. Using a result by Vangeldère (see [1], I.5.1) we have

$$^l[\,{}^i P_E(A - B)] = {}^l P_E(A - B).$$

Therefore,

1) A convex set is expansive, if ${}^i C \neq \emptyset$.

$$E_1 := {}^{li}P_E(A - B) = {}^{l}P_E(A - B) = {}^{li}P_{E_1}(A - B) = {}^{l}P_{E_1}(A - B)$$

is a subspace of E and

$$(4) \qquad 0 \in {}^{i}P_E(A - B) = {}^{i}P_{E_1}(A - B)$$

is satisfied. Then A, B and C(A - B) are subsets of $E_1 \times F$. Now we can restrict our consideration to the space $E_1 \times F$. From (4) it follows that for every $x \in E_1$ there exists $t_1 \in \in R_+$, $t_1 \neq 0$, such that for any $t \in [0,t_1)$ there are $y_1 := := y_1(t) \in F$ and $y_2 := y_2(t) \in F$ with $(tx, y_1 - y_2) \in A - B$. Then we can find such x_1 and x_2 in E_1 for which

$$(5) \quad (tx, y_1 - y_2) = (x_1 - x_2, y_1 - y_2) = (x_1, y_1) - - (x_2, y_2) \in A - B;$$

now we define

$$(6) \quad F_x := \{y \in F/(x,y) \in C(A - B)\}, \quad x \in E_1.$$

From (5) we get $t^{-1}(y_1 - y_2) \in F_x$ for $t \in (0,t_1)$. This shows

$$(7) \quad F_x \neq \emptyset \text{ for all } x \in E_1.$$

Moreover, one has

$$(8) \quad F_0 \subseteq F_+ .$$

Let $y \in F_0 \setminus \{0\}$ be fixed. Then, using (6) and the definition of C(A - B), there exist $t \in R_+$, $t \neq 0$, and points $(x_1, y_1) \in A$ and $(x_2, y_2) \in B$ such that

$$(0,y) = t[(x_1,y_1) - (x_2,y_2)], \text{ where } x_1 = x_2 .$$

By (3) one has $y_2 \leq y_1$; that means $y = t(y_1 - y_2) \in F_+$. For fixed $x, x' \in E_1$ we have

$$F_x + F_{x'} = \{y/(x,y) \in C(A - B)\} + \{y'/(x',y') \in C(A - B)\} ;$$

then for fixed $y \in F_x$ and $y' \in F_{x'}$, it holds

$$(x,y) + (x',y') \in C(A - B) + C(A - B) = C(A - B)$$

since $C(A - B)$ is a convex cone. Therefore

$$(x + x', y + y') \in C(A - B) ;$$

that means $y + y' \in F_{x+x'}$. Thus

$$(9) \quad F_x + F_{x'} \subseteq F_{x+x'} .$$

Now we are able to show that F_x has a lower bound in F for every $x \in E_1$. Let $x \in E_1$ be fixed. Then, by (7), there exists y' with $-y' \in F_{-x}$. From (9) and (8) it follows $y - y' \in F_x + F_{-x} \subseteq F_0 \subseteq F_+$ for all $y \in F_x$. Hence $y' \leqq y$ for all $y \in F_x$.

Since F has the least upper bound property, the operator T given by

$$(10) \quad T(x) := \inf \{y/y \in F_x\}$$

is well-defined for all $x \in E_1$; and one has $T: E_1 \longrightarrow F$. For this mapping we get

$$\begin{aligned} T(x + x') &= \inf \{\bar{y}/\bar{y} \in F_{x+x'}\} \\ &= \inf_{y,y'} \{y + y'/y + y' \in F_{x+x'}\} \\ &\leqq \inf_{y,y'} \{y + y'/y \in F_x,\ y' \in F_{x'}\} \\ &= \inf \{y/y \in F_x\} + \inf \{y'/y' \in F_{x'}\} \\ &= T(x) + T(x') \end{aligned}$$

for all $x, x' \in E_1$. Now let $t \in R_+$, $t \neq 0$, and $x \in E_1$ be fixed. Then

$$T(tx) = \inf\{y/y \in F_{tx}\} = \inf\{y/y \in tF_x\}$$
$$= \inf\{ty'/y' \in F_x\} = t\,\inf\{y'/y' \in F_x\}$$
$$= tT(x) .$$

This relation is true also for t = 0. Therefore, the operator T defined by (10) is sublinear.

Thus, using (HB), there exists an $L \in \mathcal{L}(E_1,F)$ such that $L(x) \leqq T(x)$ for all $x \in E_1$. Combining this with (10),(6) and the definition of the cone C(A - B) we get for $x = x_1 - - x_2$

$$L(x_1 - x_2) \leqq T(x_1 - x_2) \leqq y_1 - y_2 \quad \forall (x_1,y_1) \in A, \quad \forall (x_2,y_2) \in B.$$

Since F has the least upper bound property, this implies

$$(11)\quad L(x_1) - y_1 \leqq y_0 \leqq L(x_2) - y_2 \quad \forall (x_1,y_1) \in A, \quad \forall (x_2,y_2) \in B,$$

where $y_0 \in F$ is an element for which

$$\sup\{L(x_1) - y_1/(x_1,y_1) \in A\} \leqq y_0 \leqq \inf\{L(x_2) - y_2/(x_2,y_2) \in B\}$$

is satisfied. Let E_2 be an algebraical complementary space of E_1. Then an arbitrary $z \in E$ has a unique representation in the following way: $z = x + u$, $x \in E_1$, $u \in E_2$ (see [7], p. 54). By (11) we can see that L' defined by $L'(z) = L'(x + u) = L(x)$ for all $z \in E$ is convenient.

Conversely, it is clear that (2) implies (3).

§ 4. Equivalent conditions. In this section we consider the following properties of F using the assumption (A):

Let F(U) and D(V) be subsets of E such that

$P_o := D(U) \cap D(V)$ is nonempty, let $U:D(U) \longrightarrow E(K)$, $V:D(V) \longrightarrow$ $\longrightarrow F$ and let $C(A)$ be convex [2] for

$$(12)\quad A := \{(U(x)+k,\ V(x) + f - u)/x \in P_o,\ k \in K,\ f \in F_+\}$$

with

$$(13)\quad u := \inf \{V(x)/x \in P_o,\ -U(x) \in K\} .$$

Let $U(P_o) + K$ be an expansive set such that

$$(14)\quad 0 \in {}^{i}[U(P_o) + K] .$$

Modified Hahn-Banach extension property (MHB): Let $D(L_o)$ be a symmetric subset of E, let $D(T)$ be a subset of E such that $D(T) \supseteq D(L_o)$, $D(T) - D(L_o)$ is expansive and $0 \in {}^{i}[D(T) - D(L_o)]$. If $T:D(T) \longrightarrow F$ and $L_o:D(L_o) \longrightarrow F$ are mappings for which $C(\text{epi } T - \text{hypo } L_o)$ is a convex set and

$$(15)\quad T(0) = 0,$$

$$(16)\quad L_o(x) \leqq T(x) \qquad \forall x \in D(L_o),$$

$$(17)\quad -L_o(x) = L_o(-x) \qquad \forall x \in D(L_o)$$

are satisfied, then there exists an $L \in \mathcal{L}(E,F)$ such that

$$(18)\quad L_o(x) = L(x) \qquad \forall x \in D(L_o),$$

$$(19)\quad L(x) \leqq T(x) \qquad \forall x \in D(T).$$

Modified Farkas-Minkowski property (MFM): Under assumption (A) we have

$$(20)\quad \left.\begin{matrix} -U(x) \in K \\ x \in P_o \end{matrix}\right\} \Longrightarrow 0 \leqq V(x)$$

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2) In [8] we have given some sufficient conditions for this property.

if and only if there exists an $L \in \mathcal{L}_+(E(K),F)$ such that

$$(21)\quad 0 \not\leq V(x) + L(U(x)) \qquad \forall x \in P_o .$$

<u>Modified Kuhn-Tucker property</u> (MKT): (a) Let assumption (A) be satisfied. If x_o is a solution of problem

(P) find $x_o \in G$ with $G := \{x \in P_o / -U(x) \in K\}$ such that

$V(x_o) \not\geq V(x)$ for all $x \in G$,

then there exists an $L_o \in \mathcal{L}_+(E(K),F)$ such that (x_o, L_o) is a solution of problem

(SP) find $(x_o, L_o) \in P_o \times \mathcal{L}_+(E(K),F)$ such that

$\Phi(x_o, L) \not\geq \Phi(x_o, L_o) \not\geq \Phi(x, L_o)$ for all $x \in P_o$ and all $L \in \mathcal{L}_+(E(K),F)$, where Φ is the Lagrange-mapping defined by

$\Phi(x,L) := V(x) + L(U(x))$, $x \in P_o$, $L \in \mathcal{L}_+(E(K),F)$.

(b) If the order-cone K has the properties ${}^iK \neq \emptyset$ and $K = {}^bK$ [3] and if (x_o, L_o) is a solution of (SP), then x_o is a solution of (P).

<u>Modified Krein property</u> (MK): Let D be a nonempty symmetric convex subset of E(K), and let $L_1 : D \longrightarrow F$ be a convex mapping such that

$0 \not\leq L_1(x) \qquad \forall x \in D \cap K,$

$L_1(-x) = -L_1(x) \qquad \forall x \in D.$

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3) For a subset $K \subseteq E(K)$ we denote the algebraical hull by ${}^bK$ that means ${}^bK := K \cup {}^aK$, where ${}^aK$ contains all points of E(K) which are linear atteinable of K.

If $0 \in {}^{i}(D + K)$, then there exists an $L \in \mathcal{L}_+(E(K),F)$ such that

$$L_1(x) = L(x) \qquad \forall x \in D.$$

Krein property (K): Let A be a subspace of E(K) such that A - K is also a subspace. If $L_o \in \mathcal{L}_+(A(A \cap K),F)$, then there exists an $L \in \mathcal{L}_+(E(K),F)$ such that

$$L_o(x) = L(x) \qquad \forall x \in A.$$

Theorem 2. The properties (lub), (HB), (S), (MHB), (MFM), (MKT), (MK) and (K) are equivalent for a partially ordered vector space F.

Proof. In order to show these equivalences we prove the following implications

(S) $\Longrightarrow$ (MHB) $\Longrightarrow$ (HB),

(lub) $\Longrightarrow$ (MFM) $\Longrightarrow$ (MKT) $\Longrightarrow$ (MK) $\Longrightarrow$ (K).

It is referred to [2], p. 136, for a proof of (K) $\Longrightarrow$ (HB).

1. (S) $\Longrightarrow$ (MHB): We put A:= epi T and B:= hypo L_o. Then (16) implies (3). By (S) there exist $L \in \mathcal{L}(E,F)$ and $y_o \in F$ with

$$L(x) - y_1 \leqq y_o \leqq L(y) - y_2 \quad \forall (x,y_1) \in \text{epi } T, \; \forall (y,y_2) \in \text{hypo } L_o.$$

For $y_1 = T(x)$ and $y_2 = L_o(y)$ we get

(22) $L(x) - T(x) \leqq y_o \qquad \forall x \in D(T),$

(23) $L(y) - L_o(y) \geqq y_o \qquad \forall y \in D(L_o).$

By means of (17) and (23) one has $0 \geqq y_o$ and, therefore, (22) implies (19). Combining (15) and (22) we obtain $y_o = 0$. In view of (23) and (17) it follows (18).

2. (MHB) $\Longrightarrow$ (HB): We apply (MHB) to D(T) = E, $D(L_o)$ = A, where the mappings $T:E \longrightarrow F$, $L_o:A \longrightarrow F$ are sublinear and linear, respectively.

Therefore, in connection with Theorem 1 and To's result we have the following equivalences:

(lub) $\Longleftrightarrow$ (HB) $\Longleftrightarrow$ (S) $\Longleftrightarrow$ (MHB).

3. (lub) $\Longrightarrow$ (MFM): Let (20) be satisfied. By (lub) and (20) u defined by (13) is contained in F_+ . Moreover, $U(x) + k = 0$ with $k \in K$ implies $u \notin V(x)$ and, therefore, one has $0 \notin V(x) + f - u$ for all $f \in F_+$. Since (lub) is equivalent to (S), we are able to apply (S) to the sets $B := \{(0,0)\} \subseteq E(K) \times F$ and A defined by (12). In that way there exists $-L \in \mathcal{L}(E(K),F)$ such that

$$-L(U(x) + k) - V(x) - f + u \not\leq 0 \quad \forall x \in P_o, \ \forall k \in K, \quad \forall f \in F_+ .$$

Since $u \in F_+$, we get for $f = 0$

$$(24) \quad L(U(x) + k) + V(x) \geq u \geq 0 \qquad \forall x \in P_o, \ \forall k \in K .$$

In order to prove $L \in \mathcal{L}_+(E(K),F)$ let $x \in P_o$ and $k \in K$ be fixed elements. Then for each $t \in R_+$, $t \neq 0$, we have

$$L(U(x) + tk) + V(x) = L(U(x)) + V(x) + tL(k) \geq 0 .$$

Therefore (see [6], Lemma A), it follows

$$\inf \{L(k) + t^{-1} [L(U(x)) + V(x)] / t > 0\} = L(k) \geq 0$$

because we get from (24) for $k = 0$ (21). Hence $L \in \mathcal{L}_+(E(K),F)$.

Conversely, it is clear that (20) is a consequence of (21).

4. (MFM) $\Longrightarrow$ (MKT): Applying (MFM) to the mappings U and V' defined by

$$V'(x) := V(x) - V(x_0), \quad x \in D(V),$$

we get from (21)

$$(25) \quad L_0(U(x)) + V(x) \geqq V(x_0) \qquad \forall x \in P_0$$

for at least one $L_0 \in \mathcal{L}_+(E(K),F)$. Hence $L_0(U(x_0)) \geqq 0$.

On the other hand we have $L_0(U(x_0)) \leqq 0$ because of $U(x_0) \leqq 0$. Therefore, it is $L_0(U(x_0)) = 0$. Then (25) leads to

$$(26) \quad L_0(U(x_0)) + V(x_0) \not\geq L_0(U(x)) + V(x) \qquad \forall x \in P_0 .$$

Since $U(x_0) \not\geq 0$, one has $L(U(x_0)) \not\geq 0$ for all $L \in \mathcal{L}_+(E(K),F)$ and we get

$$L(U(x_0)) + V(x_0) \not\geq L_0(U(x_0)) + V(x_0) \qquad \forall L \in \mathcal{L}_+(E(K),F).$$

In connection with (26) (MKT), part (a), is proved. Part (b) is shown in [5].

5. (MKT) $\Longrightarrow$ (MK): It is easy to see that D + K is convex and, therefore, this set is expansive, too. If we put $E = E(K) + D(U)$, $D = D(V)$, $V = L_1$ and $U = -I$, where $I(x) = x$ for all $x \in E(K)$, then all assumptions of (MKT) are satisfied and we have $P_0 = D$,

$$G = \{ x \in P_0 / x \in K \} = D \cap K.$$

Moreover, $x_0 = 0$ is a solution of problem (P). By (MKT) then there exists $L_0 \in \mathcal{L}_+(E(K),F)$ such that

$$L_1(x_0) + L_0(-x_0) \not\geq L_1(x) + L_0(-x) \qquad \forall x \in D.$$

From this we get $L_0(x) \not\geq L_1(x)$ for all $x \in D$. That means

$$L_0(x) = L_1(x) \qquad \forall x \in D,$$

since $D = -D$ and $-L_1(x) = L_1(-x)$. Therefore, $L = L_0$ is convenient.

6. (MK) $\Longrightarrow$ (K): We choose in (MK) $D = A$, $L_1 = L_0$.

References

[1] BAIR J. et FOURNEAU R.: Etude Géométrique des Espaces Vectoriels, Lecture Notes in Mathematics 489, Springer-Verlag, Berlin-Heidelberg-New York 1975.

[2] DAY M.M.: Normed Linear Spaces, Ergebnisse der Math., Bd. 21, Springer-Verlag, Berlin-Heidelberg-New York 1973, third edition.

[3] ELSTER K.-H. und NEHSE R.: Konjugierte Operatoren und Subdifferentiale, Mathematische Operationsforsch. und Statist. 6(1975), 641-657.

[4] ELSTER K.-H. und NEHSE R.: Necessary and sufficient conditions for the order-completeness of partially ordered vector spaces, to appear in Math. Nachr.

[5] ELSTER K.-H. und NEHSE R.: Nichtkonvexe Optimierungsprobleme, 21. Intern. Wiss. Koll. TH Ilmenau 1976, Vortragsauszüge Reihe B_1, 69-72.

[6] FELDMAN M.M.: On sufficient conditions for the existence of supports of sublinear operators, Sibir. Math. J. 16(1975), 132-138, in Russian.

[7] KÖTHE G.: Topologische lineare Räume I. Springer-Verlag, Berlin-Heidelberg-New York 1966, 2. Aufl.

[8] NEHSE R.: Some general separation theorems, to appear in Math. Nachr.

[9] TO T.O.: The equivalence of the least upper bound property and the Hahn-Banach property in ordered linear spaces, Proc. Amer. Math. Soc. 30(1972), 287-295.

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