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Jiří Vinárek

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REMARKS ON SUBDIRECT REPRESENTATIONS IN CATEGORIES

Jiří VINÁREK, Praha

Abstract: Possibilities of a generalization of the Birkhoff representation theorem for concrete categories are discussed. We present some generalizations of this theorem for a certain class of categories (including e.g. relational systems, topological spaces, partially ordered sets etc.). Examples of concrete categories for which a generalization of the mentioned Birkhoff theorem is not possible are also discussed.

Key words: Subdirect irreducibility, concrete category, subobject, semiregular category, subdirect representation.

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The concept of subdirect irreducibility was introduced for algebras by G. Birkhoff in [1]. A variant of his definition making difference between subobjects and general monomorphisms (which is unnecessary with algebras) can be applied also for graphs (see [5]) and for general concrete categories (see [4],[6]). G. Birkhoff proved that every algebra of a finite type has a subdirect representation. A similar assertion holds also for finite objects of regular categories (see [4]). We are going to present examples of categories where there are objects with no subdirect representation.

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Definition. Let $(\mathfrak{K},U)$ be a concrete category, $A \subset \operatorname{obj} \mathfrak{K}$. Then A is said to have a representation in $\mathcal{A}$ if there exist objects $(A_j)_{j\in J}$, $A_j \in \mathcal{A}$, a product $\prod_{j\in J} A_j$ with projections p_j and a subobject $\mu : A \longrightarrow \prod A_j$ such that $U(p_j\mu)$ is onto for every $j \in J$.

Remark. In particular, we shall use this definition for representations in classes of subdirectly and meet irreducibles (see [4]).

First, we recall some definitions:

(a) Let $(\mathfrak{K},U)$ be a concrete category, X a set and $\mathfrak{K}UX = (\{A \in \operatorname{obj}\mathfrak{K} \mid UA = X\}, \prec)$ where $\prec$ is defined by $A \prec B$ iff there exists a $\varphi : A \longrightarrow B$ with $U\varphi = 1_{UA}$. Then an object A is meet irreducible if $A = \bigwedge_{j\in J} A_j$ (in $\mathfrak{K}UX$) implies that there exists a $j \in J$ such that $A_j = A$.

(b) A subobject in a concrete category $(\mathfrak{K},U)$ is a monomorphism $\mu : A \longrightarrow B$ such that for every $f: UC \longrightarrow UA$ for which there is a $\psi : C \longrightarrow B$ with $U\psi = U\mu \circ f$ there exists a $\varphi : C \longrightarrow A$ with $U\varphi = f$.

(c) A concrete category $(\mathfrak{K},U)$ is said to be semiregular if it has the following properties: U preserves limits; for every invertible mapping $f: X \longrightarrow UA$ there is an isomorphism φ with $U\varphi = f$; if α is an isomorphism and $U\alpha = 1_{UA}$ then $\alpha = 1_A$; every $\mathfrak{K}UX$ is a set; for every φ there is a subobject decomposition $\varphi = \mu\varepsilon$ with μ a subobject and $U\varepsilon$ onto.

(d) An object A of a concrete category is said to be subdirectly irreducible (cf. [1],[4],[6]) if for every subobject $\mu : A \longrightarrow \prod_{j \in J} A_j$ such that all $U(p_j \mu)$ are onto at least one $p_j \mu$ is an isomorphism.

<u>Propositiòn 1.</u> Let a semiregular productive $(\mathfrak{K},U)$ satisfy the following conditions:

(i) Every finite object has a representation with meet irreducibles.

(ii) For every finite object A there exists $B \vdash A$ which is maximal in $\mathfrak{K}U(UA)$.

Then every finite object of $\mathfrak{K}$ has a representation with subdirectly irreducibles (i.e. a subdirect representation).

<u>Proof</u>. Suppose the contrary. Put $n = \min \{ \text{card } UA \mid UA$ is finite, A has no subdirect representation$\}$. Obviously, $n > 1$. (If card $UA \leq 1$, A is meet irreducible, then A is subdirectly irreducible, too.)

(a) Suppose there exists a maximal A, card UA = n, with no subdirect representation. Then there is a subobject $\mu : A \longrightarrow \prod_{j \in J} A_j$ such that $U(p_j \mu)$ is onto for any $j \in J$ and $p_j \mu$ is isomorphic for no $j \in J$. By the maximality of A, card $UA_j < n$ for any $j \in J$. Every A_j is supposed to have a subdirect representation. Therefore, A has a subdirect representation which contradicts the assumption.

(b) Let A be an object with card UA = n which has no subdirect representation. According to (i) we can suppose without loss of generality that A is meet irreducible. By (a), A is not maximal and by [6] (Theorem 3.6) there is a $\varphi :$ $: A \longrightarrow B$ with card $B < n$ which can be extended to no $A' \supsetneqq A$.

We can suppose that $U\varphi$ is onto.

B has a subdirect representation. By (ii), there exists a maximal $C \nvdash A$. According to (a), C has a subdirect representation.

Define $\mu : A \longrightarrow B \times C$ such that $p_B \mu = \varphi$, $p_C \mu : A \prec C$ (p_B, p_C are projections). Then $U\mu$ is one-to-one and there exists a subobject decomposition $\mu = A \prec D \xrightarrow{\mu'} B \times C$ with μ' a subobject (see [4]). Since φ cannot be extended to a stronger structure, $D = A$ and $\mu = \mu'$ is a subobject. A has a representation in $\{B,C\}$ which have subdirect representations.

Therefore, A has a subdirect representation which is a contradiction.

<u>Remark</u>. Differently from [4], we need not the finiteness of $\mathfrak{K}UX$ for any finite X here.

<u>Example 1</u>. The condition (i) in Proposition 1 is necessary: Let $Set_{[0,1]}$ be a category with the objects (A,v) where A is a set and $0 \leq v \leq 1$, and the morphisms $(A,v) \longrightarrow$ $\longrightarrow (B,w)$ mappings from A to B if $v \leq w$ and with no morphisms $(A,v) \longrightarrow (B,w)$ if $v > w$.

If $v < 1$ then $(A,v) = \bigwedge_{v<r<1} (A,r)$. Hence, such a (A,v) is not meet irreducible and (by [4]) it is not subdirectly irreducible.

$(A,1)$ is maximal and it is subdirectly irreducible iff card $A \leq 2$. Every product of maximal objects in $Set_{[0,1]}$ is maximal and every subobject of a maximal object in $Set_{[0,1]}$ is maximal as well. Hence, no object (A,v) with $v < 1$ has a subdirect representation although for every (A,v) there is

$(A,1) \succ (A,v)$ maximal.

Example 2. The condition (ii) in Proposition 1 is necessary: Indeed, let Set_{ω_0} be a category with the objects (A,n) where A is a set and n is a positive integer, and $(1,\omega_0)$ as the terminal object, and the morphisms $f:(A,n)\longrightarrow(B,m)$ where f is a mapping from A to B and $n \leq m$.

One can see that every (A,n) is isomorphic with $(A,n+1) \times (1,n)$ and therefore for a subdirectly irreducible (A,n) we have to have card $A \leq 1$. (On the other hand, any (A,n) with card $A \leq 1$ is subdirectly irreducible.) Hence, no (A,n) with card $A \geq 2$ has a subdirect representation although every (A,n) is meet irreducible (because $\mathfrak{K}UA$ is isomorphic with ω_0 (resp. $\omega_0 + 1$) for card $A \neq 1$ (card $A = 1$)).

Proposition 2. Let a semiregular productive $(\mathfrak{K},U)$ with a two-point cogenerator satisfy the following conditions:

(i) Every object of $\mathfrak{K}$ has a representation with meet irreducibles.

(ii) For every object A there exists an object $M \succ A$ which is maximal.

(iii) For every non-maximal meet irreducible B there exists a subdirectly irreducible D and a $\varphi : B \longrightarrow D$ which cannot be extended to an object $E \precneqq B$.

Then every object of $\mathfrak{K}$ has a subdirect representation.

Proof. (a) If M is maximal, card $UM \leq 2$, then one can easily see that M is subdirectly irreducible.

(b) If M is maximal, card UM > 2, C is a cogenerator, then card UC = 2 and for $(\mu_j : M \longrightarrow C)_J$ the system of all the morphisms from M to C there exists a subobject $\mu : M \rightarrow$ $\rightarrow C^J$ defined by $p_j \mu = \mu_j$. According to (a) M has a subdirect representation.

(c) Let A be non-maximal meet irreducible. According to (iii) there exists a subdirectly irreducible D and a $\varphi :$ $: B \longrightarrow D$ which cannot be extended to an $E \subsetneqq A$. Let $M \succ A$ be maximal; define $\mu : A \longrightarrow M \times D$ by $p_M \mu = A \prec M$, $p_D \mu = \varphi$ (p_M, p_D are projections). Then $U\mu$ is one-to-one and (see [4]) there is a subobject decomposition $\mu = \mu' \varepsilon$ with μ' a subobject and $\varepsilon : A \prec A'$. By the assumption, $A = A'$ and $\mu =$ $= \mu'$ is a subobject. Consequently by (a) and (b) A has a subdirect representation.

(d) According to (i),(a),(b) and (c) every object has a subdirect representation.

Remark. By Proposition 2, every object has a subdirect representation e.g. in the following categories: relational systems (in particular, directed graphs, symmetric graphs), hypergraphs, topological spaces, preordered sets, partially ordered sets etc.

Example 3. The condition (iii) in Proposition 2 is necessary. Indeed, define F: Set $\longrightarrow$ Set as follows:

$$FA = \{ X \subset A \mid \text{card } X = \omega_0 \} \cup \{ 0_A \} ,$$

and if f: $A \longrightarrow B$ then define $F(f): FA \longrightarrow FB$ putting $F(f)(0_A) = 0_B$, $F(f)(X) = f(X)$ if card $f(X) = \omega_0$, $F(f)(X) =$ $= 0_B$ otherwise.

Then the category S(F) (whose objects are couples (A,r) with A a set and $r \subset FA$ and whose morphisms $(A,r) \longrightarrow$ $\longrightarrow (B,s)$ are mappings satisfying $F(f)(r) \subset s$) has a two-point cogenerator (2,F2), satisfies (i) and (ii) and contains objects with no subdirect representation.

<u>Proof</u>. One can prove (see [6], 4.4) that S(F) has the following subdirectly irreducibles: $(X,\emptyset)$ with card $X \leq 1$, (X,FX) with card $X \leq 2$ and $(X, FX \setminus \{Y\})$ with $Y \in FX$, card $(X \setminus Y) \leq 1$. An object $(X, FX \setminus \{0_X\})$ with an infinite X has no subdirect representation (see [6], 7.2).

On the other hand, any object is either maximal - i.e. (X,FX), or it has a representation with meet irreducibles

$$(X,r) = \bigwedge_{u \in FX \setminus r} (X, FX \setminus \{u\});$$

$(X,r) \prec (X,FX)$ for every X. Thus, the conditions (i) and (ii) hold (while (iii) does not).

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Matematicko-fyzikální fakulta
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

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