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CONVEX COMBINATIONS OF COMMUTING AFFINE OPERATORS

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Abstract: Let E be a complete Hausdorff locally convex topological vector space and let R and S be two commuting mean stable affine operators on E such that the transformations R^n and S^n, $n \ge 1$, are equicontinuous on E. Under these circumstances, convex combinations of R and S are shown to be mean stable. This is a generalization of a result due to Sine, who examined linear contraction operators on a Banach space.

Key words: Affine operators, ergodic averages, mean stable, convex combinations, locally convex spaces.

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1. Introduction and the theorem. Let E be a complete Hausdorff locally convex (real or complex) topological vector space (t.v.s) and R an affine operator on E. Thus

$$R(ax + (1-a)y) = aR(x) + (1-a)R(y)$$

for all $0 < a < 1$ and all x and y in E. We call R mean stable if for every $x \in E$ the ergodic averages

$$(1/n)\sum_{i=0}^{n-1} R^i(x)$$

converge to a point of E invariant under R. In [6] Sine showed, in essence, that a convex combination of two commuting mean stable linear contraction operators on a Banach space is mean stable. Although his argument given there can easily be modified to obtain the same result provided that the hypothesis of being contraction operators is replaced by the hypothesis of being power-bounded operators, it would seem worth showing that the same result holds under a more general setting. This is the starting point of the present work.

Our result is as follows.

Theorem. Let R and S be two commuting mean stable affine operators on a complete Hausdorff locally convex (real or complex) t.v.s. E such that the transformations R^n and S^n, $n \geq 1$, are equicontinuous on E. Then for any $0 < a < 1$ the affine operator $T = aR + (1-a)S$ is mean stable, and further for every $x \in E$, $T(x) = x$ if and only if $R(x) = S(x) = x$.

It is interesting to note that in a recent congress of the Mathematical Society of Japan I learned, without proof, from Mr. K. Anzai of Keio University that he also obtained a similar result for commuting linear operators.

2. Proof of the theorem. Putting $r = R(0)$ and $s = S(0)$, it is easily seen that the two mappings A and B defined by

$$A(x) = R(x) - r \quad \text{and} \quad B(x) = S(x) - s \qquad (x \in E)$$

are (real) linear operators on E. Let $p \in E$ be such that $R(p) = p$. Then $RS^n(p) = S^nR(p) = S^n(p)$ for all $n \geq 0$, and thus if we let

$$q = \lim_n (1/n) \sum_{i=0}^{n-1} S^i(p),$$

then $R(q) = S(q) = q$. Therefore for all $x \in E$ we have

$$R(x) = A(x-q) + A(q) + r = A(x-q) + q,$$

$$S(x) = B(x-q) + q,$$

and

$$T(x) = C(x-q) + q, \text{ where } C = aA + (1-a)B.$$

From this it follows directly that A and B are mean stable (real) linear operators on E, that A^n and B^n, $n \geq 1$, are equicontinuous on E, and that $AB = BA$. It is now enough to show that C is mean stable and that for every $x \in E$, $C(x) = x$ if and only if $A(x) = B(x) = x$.

First let us show that $C(x) = x$ implies $A(x) = B(x) = x$. (The converse implication is obvious.) To do this, let U be any convex neighborhood of $0 \in E$, and choose a neighborhood V of $0 \in E$ so that

$$A^m B^n(V) \subset U \quad \text{for all} \quad m, n \geq 0.$$

Denote by W the convex hull of $\cup \{ A^m B^n(V) : m, n \geq 0 \}$. Since U is convex, it follows that

$$W \subset U,$$

and furthermore we have

$$A(W) \subset W \quad \text{and} \quad B(W) \subset W.$$

Let $\| \cdot \|$ denote the Minkowski functional of W and $N = \{ z : \| z \| = 0 \}$. Since $\| z \| < 1$ implies $\| A(z) \| < 1$ and $\| B(z) \| < 1$, if we set

$$A'(z+N) = A(z) + N,$$

$$B'(z+N) = B(z) + N,$$

and

$$C'(z+N) = C(z) + N,$$

then A', B' and C' are (real) linear operators on the quotient (normed) space E/N such that $\|A'\| \leq 1$, $\|B'\| \leq 1$ and $C' = aA' + (1-a)B'$. Clearly $C(x) = x$ implies $C'(x+N) = x+N$ and hence, by Lemma 1 of Falkowitz [2], we have

$$A'(x+N) = B'(x+N) = x + N.$$

Therefore

$$A(x) - x \in N \subset W \subset U \quad \text{and} \quad B(x) - x \in U.$$

This shows that $A(x) = B(x) = x$.

For a continuous (real) linear operator Q on E, we shall define $F(Q) = \{x \in E;\ Q(x) = x\}$ and $F^*(Q^*) = \{x^* \in E^* : Q^*(x^*) = x^*\}$, where E^* denotes the topological dual of E and Q^* denotes the adjoint of Q. (We may and will assume, without loss of generality, in the proof that E is a real t.v.s.)

To prove that C is mean stable, we use the results of the author [4] and understand that it is sufficient to show that $F(C)$ separates points of $F^*(C^*)$.

To do this, let $x^* \in F^*(C^*)$ and $x^* \neq 0$, and choose a neighborhood U of $0 \in E$ so that

$$|\langle x, x^* \rangle| \leq 1 \quad \text{for all } x \in U.$$

Further choose another neighborhood V of $0 \in E$ so that

$$A^m B^n(V) \subset U \quad \text{for all } m, n \geq 0.$$

Write

$$W = \bigcup\{ A^m B^n(V) : m, n \geq 0 \}$$

and

$$K = \{ z^* \in E^* : |\langle x, z^* \rangle| \leq 1 \text{ for all } x \in W \}.$$

It is clear that

$$x^* \in K, \quad A^*(K) \subset K \quad \text{and} \quad B^*(K) \subset K,$$

and by the Banach-Alaoglu theorem (cf. [3], Theorem 3.15), K is a weak*-compact convex subset of E^*. Let

$$K(C^*) = K \cap F^*(C^*),$$

and let $y^* \in K(C^*)$ be any extreme point of $K(C^*)$. We then have, as in Sine [6], that

$$C^* A^*(y^*) = A^* C^*(y^*) = A^*(y) \quad \text{and}$$

$$C^* B^*(y^*) = B^*(y^*);$$

hence $A^*(y^*) = B^*(y^*) = y^*$, because $y^* = C^*(y^*) = aA^*(y^*) + (1-a)B^*(y^*)$. This and the Krein-Milman theorem (cf. [3], Theorem 3.21) imply that

$$K(C^*) \subset K \cap F^*(A^*) \cap F^*(B^*);$$

in particular, $x^* \in F^*(A^*) \cap F^*(B^*)$. Since A is mean stable if and only if F(A) separates points of $F^*(A^*)$, by the results of [4], then there exists a point $x \in F(A)$ satisfying $\langle x, x^* \rangle \neq 0$. Since AB = BA implies $B^n(x) \in F(A)$ for all $n \geq 0$, if we let

$$y = \lim_n (1/n) \sum_{i=o}^{n-1} B^i(x),$$

then $y \in F(A) \cap F(B) \subset F(C)$, and further $\langle y,x^* \rangle = \langle x,x^* \rangle \neq 0$.

This establishes the theorem.

3. Remarks. Throughout this section, R will be an affine operator on a complete Hausdorff locally convex (real) t.v.s. E such that the transformations R^n, $n \geq 1$, are equicontinuous on E. Define

$$F = \{x \in E : R(x) = x\}$$

and

$$F^* = \{x^* \in E^* : \langle R(x),x^* \rangle = \langle x,x^* \rangle \quad \text{for all } x \in E\} .$$

It is easily seen (cf. the preceding section) that if $F \neq \phi$ then F is a closed affine subspace of E, i.e., F has the form $F = p + D$ where $p \in E$ and D is a closed linear subspace of E, and that F^* is a weak*-closed linear subspace of E^*.

(1) Suppose $F \neq \phi$. Then R is mean stable if and only if any $x^* \in F^*$, with $x^* \neq 0$, is not constant on F.

To see this, fix any point $p \in F$. As in the preceding section, we have

$$R(x) = A(x-p) + p \qquad (x \in E)$$

where A is a (real) linear operator on E. It follows that R is mean stable if and only if A is mean stable. By [4], the latter condition is equivalent to the fact that $F(A)$ separates points of $F^*(A^*)$. Since $F = p + F(A)$ and $F^* = F^*(A^*)$ by an easy observation, this completes the proof of (1).

2) Suppose $x \in E$ and the set $\{R^n(x) : n \geq 0\}$ is relatively weakly compact in E. Then $F \neq \phi$.

To see this, let K denote the closed convex hull of the set $\{R^n(x): n \geq 0\}$. By Krein's theorem (cf. [5], Theorem IV.11.4), K is again weakly compact. Since R is weakly continuous and $R(K) \subset K$, it follows from the Markov-Kakutani fixed point theorem (cf. [1], Theorem V.10.6) that there exists a point p in $K \subset E$ such that $R(p) = p$. This establishes (2).

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