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$L_f(a,r)$-SPACES BETWEEN WHICH ALL THE OPERATORS ARE COMPACT, I

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Abstract: Certain couples of $L_f(a,r)$-spaces, between which all the diagonal operators are compact, are characterized. It turns out that the result is "symmetric", i.e. "all the diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ are compact if and only if all the diagonal operators from $L_g(b,s)$ to $L_f(a,r)$ are compact".

Key words: Nuclear Fréchet space, compact operator, diagonal operator.

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§ 1. Introduction. The investigation, started in this paper, has to be considered as being part of the general investigation on the structure of (concrete) nuclear Fréchet spaces, started by Dragelev in [2]. One way to obtain information about the structure of particular spaces is to consider the types of operators between them. The nuclear Fréchet spaces we are interested in, belong to the class of $L_f(a,r)$-spaces, introduced in [2]. We want to characterize in fact certain couples of $L_f(a,r)$-spaces between which all the operators are compact. At the moment we restrict ourselves to the diagonal operators. The general case will be treated in

a forthcoming paper (part II).
The definition of an $L_f(a,r)$-space as well as the basic properties determining the nature of the couples of spaces to be considered, are given in § 2. The main results are proved in § 4. For convenience, we collect some preliminary lemmas in § 3.

The following notions and basic properties will be used throughout the paper without further reference.

1. Let P be a sequence of strictly positive sequences $a^k = (a_n^k)$ such that $a^k < a^{k+1}$ for all k. The Köthe-space $\Lambda(P)$ is then defined as

$$\Lambda(P) = \bigcap_k \frac{1}{a^k} \cdot \ell^1 = \{ (\gamma_n) \mid \| (\gamma_n) \|_k = \Sigma_n |\gamma_n| \, a_n^k < \infty ,$$
$$k = 1,2,\dots \}$$

and topologized by the sequence of norms $\| \cdot \|_k$, $k = 1,2,\dots$.
$\Lambda(P)$ is a perfect Fréchet sequence space. Its topological dual space is the space $\Lambda(P)^\times = \bigcup_k a^k \cdot \ell^\infty$ (see [3]).

2. It follows from the Grothendieck-Pietsch criterion (see [4]) that $\Lambda(P)$ is nuclear if and only if the sequence of norms $(\| \cdot \|_k)$ on $\Lambda(P)$ is equivalent to the sequence of norms $\| (\gamma_n) \|_k' = \sup_n |\gamma_n| \, a_n^k$.

3. If I is a subsequence of IN (the natural numbers) then the sequence space $\Lambda(P)_I = \{ (\gamma_n) \mid (\gamma_n) \in \Lambda(P),\ \gamma_n = 0 \text{ if } n \notin I) \}$ is called a step-space of $\Lambda(P)$.
We denote by i_I (resp. p_I) the corresponding canonical injection (resp. projection).

4. A map $T: \Lambda(P) \longrightarrow \Lambda(Q)$ is called a generalized diagonal operator if there exist subsequences I and J of $\mathbb{N}$ and

a diagonal map $T_o: \Lambda(P)_I \longrightarrow \Lambda(Q)_J$ such that $i_J \circ T_o \circ p_I =$
$= T$.

5. Let (α_n) be an increasing sequence of positive numbers such that $\lim \alpha_n = \infty$.

For $P = \{ (k^{\alpha_n})_n \mid k = 1,2,\dots \}$, the corresponding Köthe-space is called a power series space of infinite type.

For $P = \{ ((\frac{k}{k+1})^{\alpha_n})_n \mid k = 1,2,\dots \}$, the corresponding Köthe-space is called a power series space of finite type.

6. Let $\Lambda(P)$ and $\Lambda(Q)$ be nuclear Köthe-spaces and T a continuous linear map from $\Lambda(P)$ to $\Lambda(Q)$. Then

i) For each m there exists a k such that

$$\sup_j \frac{\| T(e_j) \|_m}{\| e_j \|_k} < \infty$$

(by e_j we mean the j^{th} coordinate vector).

ii) T is compact if and only if there exists a k such that for all m

$$\sup_j \frac{\| T(e_j) \|_m}{\| e_j \|_k} < \infty$$

(see [1] (2.1)).

7. A function f defined on $(0,\infty($ is called logarithmically convex if the function $\varphi(w) = \log f(e^w)$ is convex on $)-\infty, \infty($.

8. Let f be an increasing, logarithmically convex function defined on $(0,\infty($.

Then for every $a > 1$, $\lim_{u \to \infty} \frac{f(au)}{f(u)} = \tau(a)$ exists. Moreover either $\tau(a) = \infty$ for all $a > 1$ ($*$), or $\tau(a) < \infty$ for all $a > 1$ ($**$). In this case $\tau(a)$ increases to ∞ if a increases to ∞. (See [6] Lemma 7.)

A function having property ($*$)(resp. ($**$)) will be called rapidly (resp. slowly) increasing.

9. If E and F are locally convex spaces such that all the operators from E to F are compact, we'll write $(E,F) \in \mathcal{R}$. (This notation was introduced in [6].)

§ 2. $L_f(a,r)$-spaces

a. Definition ([2] p. 77). Let f be an increasing, odd function, defined on $)-\infty, \infty($, which is logarithmically convex on $(0, \infty($.

Let (a_n) be an increasing sequence of positive numbers such that $\lim a_n = \infty$.

Let (r_p) be an increasing sequence of real numbers with $\lim\limits_p r_p = r$, where $-\infty < r \leq \infty$.

Put $P = \{ (e^{f(r_p a_n)})_n \mid p = 1,2,\ldots \}$.

Then the corresponding Köthe-space $\Lambda(P)$ is called an $L_f(a,r)$-space.

I.e.

$$L_f(a,r) = \{ (\gamma_n) \mid \|(\gamma_n)\|_p = \Sigma_n |\gamma_n| \, e^{f(r_p a_n)} < \infty ,$$
$$p = 1,2,\ldots \}$$

(remark that if $r = s = \infty$ (resp. $r = s = 0$) we can putt $r_k = k$ (resp. $r_k = -k^{-1}$) for $k = 1,2,\ldots$).

In this paper we only consider nuclear $L_f(a,r)$-spaces.

Necessary and sufficient conditions for an $L_f(a,r)$-space to be nuclear can be found in [2] p. 78.

b. For what follows, the next properties are fundamental:

(1) [2], p. 79: If f and g are rapidly increasing, the spaces $L_f(a,r)$ and $L_g(b,s)$ can be isomorphic, only in the following cases:

a) $0 < r,\ s < \infty$; b) $-\infty < r,s < 0$; c) $r = s = \infty$;

d) $r = s = 0$.

(2) [2] p. 79: A space $L_f(a,r)$ is isomorphic to a power series space if and only if f is slowly increasing. In that case $L_f(a,r)$ is isomorphic to a power series space of finite (resp. infinite) type if $-\infty < r \leq 0$ (resp. $0 < r \leq \infty$).

(3) [2] p. 80: If $f^{-1} \circ g$ is rapidly increasing the spaces $L_f(a,r)$ and $L_g(b,s)$ cannot be isomorphic. (It is proved in fact in [6] p. 211 that in this case $(L_f(a,r), L_g(b,s)) \in \in \mathcal{R}$.)

c. From properties (b.1) and (b.2) the following disjoint classes of $L_f(a,r)$-spaces, with the property that two spaces belonging to different classes cannot be isomorphic, are obtained in a natural way (cfr. [2] p. 81).

We'll say that $L_f(a,r)$ and $L_g(b,s)$ belong to the same class if one of the following conditions is satisfied:

A. A_1: f and g are slowly increasing and $0 < r,s \leq \infty$.

A_2: f and g are slowly increasing and $-\infty < r,s \leq 0$.

B. B_1: f and g are rapidly increasing and $0 < r,s < \infty$.

B_2: f and g are rapidly increasing and $r = s = \infty$.

B_3: f and g are rapidly increasing and $-\infty < r,s < 0$.

B_4: f and g are rapidly increasing and $r = s = 0$.

d. It turns out that in many cases all the operators between spaces belonging to different classes are compact (see [6]). The problem we are interested in, is to characterize within each class those couples of spaces between which all the operators are compact. It follows from property (b.3) that this problem is really meaningfull only when we make the extra assumption that $f^{-1}\circ g$ and $g^{-1}\circ f$ are slowly increasing (°). The problem considered here has been solved completely for the classes A_1 and A_2 in [1]. It is in fact completely solved for the whole class A (see [5]). The solution of the problem for the other classes is the subject of our investigation. The main result in this paper shows that the relation "All the diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ are compact" is symmetric.

e. Finally we recall what is known about the isomorphism of spaces belonging to the same class (see [2] p. 79).

1. $L_f(a,r)$ and $L_f(b,s)$ are isomorphic if and only if:

(a) both spaces belong to the class B_2 or B_4 and

$$0 < \underline{\lim}\, \frac{a_n}{b_n} \leq \overline{\lim}\, \frac{a_n}{b_n} < \infty$$

or

(b) both spaces belong to the class B_1 or B_3 and

$$\lim \frac{a_n\, r}{b_n\, s} = 1$$

2. $L_f(a,r)$ is isomorphic to $L_g(b,s)$ if:

(a) both spaces belong to the class B_1 or B_3 and

$f(ra_n) = g(s\ b_n)$, for n sufficiently large,

or

(b) both spaces belong to the class B_2 or B_4 and

$f(a_n) = g(b_n)$, for n sufficiently large.

The results under a.2) are improved in Lemma 2, § 3.

Throughout the paper we'll assume that in $L_f(a,r)$ <u>the function</u> f <u>is rapidly increasing</u>. The extra assumption ($^{\circ}$) (§ 2, d.) will be mentioned only when necessary.

§ 3. <u>Preliminary Lemma's</u>

<u>Lemma 1</u>. Consider the statement:

(∗) $\exists k_1$ such that $\forall m$, $\exists n_m$ with $g(s_m b_n) < f(r_{k_1} a_n)$, for $n \geq n_m$.

Then

(i) (∗) is equivalent to

(∗∗) $\exists k_2$, $\exists n_o$ such that $g(sb_n) < f(r_{k_2} a_n)$, for $n \geq n_o$, whenever $0 < r$, $s < \infty$ and $f^{-1} \circ g$ is slowly increasing.

(ii) (∗) is equivalent to

(∗∗∗) $\forall_m$, $\exists n_m$ such that $g(mb_n) < f(a_n)$, for $n \geq n_m$, whenever $r = s = \infty$ and $g^{-1} \circ f$ is slowly increasing.

<u>Proof</u>: <u>case (i)</u>: We only need to prove that (∗) implies (∗∗). So fix $k_2 > k_1$, then there exist m_1 and n_1 such that:

$$\frac{f^{-1} \circ g(s_{m_1} b_n)}{f^{-1} \circ g(s\ b_n)} > \frac{r_{k_1}}{r_{k_2}}, \text{ for } n \geq n_1.$$

Put $n_o = \max(n_1, n_{m_1})$. Then we have for $n > n_o$:

$$r_{k_2} a_n = \frac{r_{k_2}}{r_{k_1}} r_{k_1} a_n > \frac{r_{k_2}}{r_{k_1}} f^{-1} \circ g(s_{m_1} b_n) > f^{-1} \circ g(s\, b_n),$$

which gives (**).

<u>case (ii)</u>: We prove that (*) implies (***).
There exists $t > 1$ such that

$$\frac{g^{-1} \circ f(k_1 a_n)}{g^{-1} \circ f(a_n)} < t.$$

For every m we obtain from (*) an index n_m such that:

$$(tm) b_n < g^{-1} \circ f(k_1 a_n), \text{ for } n \geq n_m$$

So $\forall m$, $\exists n_m$ such that

$$m\, b_n < \frac{1}{t} g^{-1} \circ f(k_1 a_n) < g^{-1} \circ f(a_n), \text{ for } n \geq n_m,$$

from which (***) follows.

<u>Lemma 2</u>. Suppose $f^{-1} \circ g$ and $g^{-1} \circ f$ are slowly increasing. Then $L_f(a,r) = L_g(b,s)$ and the identity operator is an isomorphism, in each of the following two cases:

(a) $0 < r, s < \infty$ and

(i) $\forall k$, $\exists n_k$ such that $f(r_k a_n) < g(s\, b_n)$, for $n \geq n_k$

and

(ii) $\forall m$, $\exists n_m$ such that $g(s_m b_n) < f(r\, a_n)$, for $n \geq n_m$.

(b) $r = s = \infty$ and

(i) $\exists k_1$, $\exists n_1$ such that $g(b_n) < f(r_{k_1} a_n)$, for $n \geq n_1$

and

(ii) $\exists m_2$, $\exists n_2$ such that $f(a_n) < g(m_2 b_n)$, for $n \geq n_2$.

<u>Proof</u>: <u>Case (a)</u>: For every m there exist k_m and n_m

such that

$$\frac{g^{-1}\circ f(r\, a_n)}{g^{-1}\circ f(r_{k_m} a_n)} < \frac{s_{m+1}}{s_m} \text{ for } n \geq n_m.$$

Hence, by (ii), we obtain for $n \geq N_m = \max(n_{m+1}, n_m)$:

$$s_{m+1} b_n < g^{-1}\circ f(r\, a_n) < g^{-1}\circ f(r_{k_m} a_n)\frac{s_{m+1}}{s_m} .$$

So, for every m there exist k_m and N_m such that

$$g(s_m b_n) < f(r_{k_m} a_n), \text{ for } n \geq N_m.$$

Hence, $L_f(a,r) \subset L_g(b,s)$ and since $\forall m, \exists k_m$

such that $\sup\limits_m e^{g(s_m b_n) - f(r_{k_m} a_n)} < \infty$, the canonical injection I: $L_f(a,r) \longrightarrow L_g(b,s)$ is continuous.

Similarly we conclude from (i) that for every k there exist m_k and n_k such that

$$f(r_k a_n) < g(s_{m_k} b_n), \text{ for } n \geq n_k.$$

This implies that $L_f(a,r) = L_g(b,s)$ and that the operator I^{-1} is continuous as well.

<u>Case (b)</u>: This is proved in the same way, making use of the inequalities:

$\forall m, \exists m_0$ such that $f^{-1}\circ g(m\, b_n) \leq m_0\, f^{-1}\circ g(b_n) < m_0 k_1 a_n$

and

$\forall k, \exists k_0$ such that $g^{-1}\circ f(k\, b_n) \leq k_0 m_1 a_n$.

§ 4. <u>Compact diagonal operators</u>

<u>Proposition 1</u>. Suppose $0 < r, s \leq \infty$.

If one of the following conditions is satisfied:

(a) $\exists\, m_1$ such that $\forall k$, $\exists\, n_k$ with $f(r_k a_n) < g(s_{m_1} b_n)$, for $n \geq n_k$, or

(b) $\exists\, k_1$ such that $\forall m$, $\exists\, n_m$ with $g(s_m b_n) < f(r_{k_1} a_n)$, for $n \geq n_m$, or

(c) there exists a partition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ of the indices such that (a) holds on $\mathbb{N}_1$ and (b) holds on $\mathbb{N}_2$,

then all the diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ (and by symmetry also from $L_g(b,s)$ to $L_f(a,r)$) are compact.

<u>Proof</u>: Let the diagonal operator $T: L_f(a,r) \longrightarrow L_g(b,s)$ be represented by the sequence (γ_n). I.e. $T((\alpha_n)) = (\alpha_n \gamma_n)_n$.

The continuity of T is expressed by:

$$(*) \quad \forall m,\ \exists\, k_m \text{ such that } \sup_{n} |\gamma_n|\, e^{g(s_m b_n) - f(r_{k_m} a_n)} < \infty .$$

The operator T will be compact iff

$$(**) \quad \exists\, k \text{ such that } \forall m: \sup_{n} |\gamma_n|\, e^{g(s_m b_n) - f(r_k a_n)} < \infty .$$

<u>Sufficiency of (a):</u> We prove that from (a) it follows that $(**)$ is satisfied for any value of k. So we fix a $k = k_2$ and take any m. Then choose $m_0 > \max(m, m_1)$, where m_1 is taken from (a).

From $(*)$ we obtain k_0 and M such that

$$|\gamma_n| < e^{M - g(s_{m_0} b_n) + f(r_{k_0} a_n)} .$$

Hence $(**)$ will be satisfied if

$$\sup_{n} e^{M - g(s_{m_0} b_n) + f(r_{k_0} a_n) + g(s_m b_n) - f(r_{k_2} a_n)} < \infty ,$$

or if $\exists\, n_m$ such that

$$\frac{M}{g(s_{m_o}b_n)} - 1 + \frac{f(r_{k_o}a_n)}{g(s_{m_o}b_n)} + \frac{g(s_m b_n)}{g(s_{m_o}b_n)} - \frac{f(r_{k_2}a_n)}{g(s_{m_o}b_n)} < 0,$$

for $n \geq n_m$.

Now $\lim_{m} \frac{M}{g(s_{m_o}b_n)} = \lim_{m} \frac{g(s_m b_n)}{s(s_{m_o}b_n)} = 0$, while $\frac{f(r_{k_2}a_n)}{g(s_{m_o}b_n)} > 0$,

for all n.

Moreover we obtain from (a):

$\exists\, n_k$ such that $0 < \frac{f(r_{k_o}a_n)}{g(s_{m_o}b_n)} < \frac{g(s_{m_1}b_n)}{g(s_{m_o}b_n)}$ for $n \geq n_k$.

Since $\lim_{m} \frac{g(s_{m_1}b_n)}{g(s_{m_o}b_n)} = 0$, the conclusion follows.

Sufficiency of (b): Choose m_o, take k_o from $(*)$ and k_1 from b). Take further $k_2 > \max(k_o, k_1)$.

An argument similar to the one used in (a) shows that for all m there exists n_m such that

$$\frac{M}{f(r_{k_2}a_n)} - \frac{g(s_{m_o}b_n)}{f(r_{k_2}a_n)} + \frac{f(r_{k_o}a_n)}{f(r_{k_2}a_n)} + \frac{g(s_m b_n)}{g(r_{k_2}a_n)} - 1 < 0,$$

for $n \geq n_m$, which implies that $(**)$ is satisfied for $k \geq k_2$.

The sufficiency of (c) follows from (a) and (b) considering the appropriate step-spaces.

Proposition 2. Suppose $-\infty < r,\ s \leq 0$.

If one of the following conditions is satisfied:

(a) $\exists\, k_1$ such that $\forall m$, $\exists\, n_m$ with $f(r_{k_1}a_n) < g(s_m b_n)$, for

$n \geq n_m$ or

(b) $\exists\, m_1$ such that $\forall k$, $\exists\, n_k$ with $g(s_{m_1} b_n) < f(r_k a_n)$, for $n \geq n_k$, or

(c) there exists a partition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ of the indices such that (a) holds on $\mathbb{N}_1$ and (b) holds on $\mathbb{N}_2$,

then all the diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ are compact.

The proof of this proposition is similar to the proof of Proposition 1 and is therefore omitted.

Remark 1. It turns out that all the results obtained in the cases $0 < r, s \leq \infty$ have, as above, their analogue in the cases $-\infty < r, s \leq 0$.

Therefore, from now on, we shall restrict our investigation to the cases $0 < r, s < \infty$ and $r = s = \infty$.

Theorem 1: If $f^{-1} \circ g$ and $g^{-1} \circ f$ are slowly increasing, then the following are equivalent:

(i) All the diagonal operators from $L_f(a,\infty)$ to $L_g(b,\infty)$ are compact

(ii) All the diagonal operators from $L_g(b,\infty)$ to $L_f(a,\infty)$ are compact

(iii) One of the following conditions:

(a) $\forall k$, $\exists\, n_k$ such that $f(ka_n) \leq g(b_n)$, for $n \geq n_k$

or

(b) $\forall m$, $\exists\, n_m$ such that $g(mb_n) \leq f(a_n)$, for $n \geq n_m$

or

(c) There exists a partition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ of the indices such that (a) holds on $\mathbb{N}_1$ and (b) holds on $\mathbb{N}_2$.

Proof: (i) $\Longrightarrow$ (iii): Put $I = \{n \mid f(a_n) \leq g(b_n)\}$ and $J = \{n \mid g(b_n) < f(a_n)\}$. Suppose J is finite, then I is infinite and we have trivially:

$$\exists\, m_1 \text{ such that } f(a_n) < g(m_1 b_n), \text{ for } n \in I.$$

Suppose there were an infinite subsequence $I' \subset I$ such that

$$\exists\, k_1 \text{ such that } g(b_n) < f(k_1 a_n), \text{ for } n \in I'.$$

Then by Lemma 2.b) § 3, the spaces $L_f(a,\infty)_I$ and $L_g(b,\infty)_I$ would be isomorphic and there would be a non-compact diagonal operator between them. This operator would then extend to a non-compact diagonal operator from $L_f(a,\infty)$ to $L_g(b,\infty)$, contradicting i). So for all k there exists an index n_k such that $g(b_n) \geq f(ka_n)$ for $n \geq n_k$. I.e. condition (a) holds. Similarly (b) holds whenever I is finite. If both I and J are infinite then (c) holds.

(iii) $\Longrightarrow$ (i): This follows immediately from Lemma 1 § 3, combined with prop. 1 § 4

(ii) $\Longrightarrow$ (i): This follows from the symmetry of the conditions in (iii).

Theorem 2: If $f^{-1} \circ g$ and $g^{-1} \circ f$ are slowly increasing then the following are equivalent:

(i) All the diagonal operators from $L_f(a,r)$ to $L_g(b,s)$ are compact

(ii) All the diagonal operators from $L_g(b,s)$ to $L_f(a,r)$ are compact

(iii) One of the following conditions:

(a) $\exists\, m_1$, $\exists\, n_1$ such that $f(ra_n) < g(s_{m_1} b_n)$ for $n \geq n_1$

or

(b) $\exists k_1$, $\exists n_1$ such that $g(sb_n) < f(r_{k_1}a_n)$ for $n \geq n_1$

or

(c) There exists a partition $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2$ of the indices such that (a) holds on $\mathbb{N}_1$ and (b) holds on $\mathbb{N}_2$.

Proof: We only need to prove (i) $\Longrightarrow$ (iii), the rest goes as in Theorem 1. Put $I = \{n \mid f(ra_n) \leq g(sb_n)\}$ and $J = \{n \mid g(sb_n) < f(ra_n)\}$.

We only treat explicitly the case where J is finite. Suppose there was a subsequence $I' \subset I$ such that

$$\lim_{n \in I'} \frac{g^{-1}f(ra_n)}{sb_n} = 1.$$

Then we would have for all m:

$$\lim_{n \in I'} \frac{g^{-1}f(ra_n)}{s_m b_n} = \frac{s}{s_m} > 1,$$

which gives:

$\forall m$, $\exists n_m$ such that $f(ra_n) > g(s_m b_n)$, for $n \geq n_m$, $n \in I'$.

Since we also trivially have:

$\forall k$; $f(r_k a_n) < g(sb_n)$, for all $n \in I'$,

we can apply Lemma 2.a) § 3 in order to obtain a non-compact diagonal operator from $L_f(a,r)$ to $L_g(b,s)$. This contradicts (ii).

We therefore must have:

$$\overline{\lim_{n \in I}} \frac{g^{-1}f(ra_n)}{sb_n} = t < 1.$$

Choose m_1 such that $t < \frac{s_{m_1}}{s}$, then $\exists n_1$ such that

$$\frac{g^{-1}f(ra_n)}{sb_n} < \frac{s_{m_1}}{s}\ , \text{ for } n \geq n_1,\ n \in I.$$

So $\exists\, m_1$ and $\exists\, n_1$ such that $f(ra_n) < g(s_{m_1} b_n)$, for $n \geq n_1$, $n \in I$, from which condition (a) in (iii) trivially follows.

Corollary 1: All the diagonal operators from $L_\varphi(a,\infty)$ to $L_\varphi(b,\infty)$ are compact if and only if one of the following conditions is satisfied:

(a) $\lim \frac{a_n}{b_n} = 0$, or b) $\lim \frac{b_n}{a_n} = 0$, or c) there exists a partition $\mathbb{N} = \mathbb{N}_1 \setminus \mathbb{N}_2$ of the indices such that

$$\lim_{n \in \mathbb{N}_1} \frac{a_n}{b_n} = 0 \text{ and } \lim_{n \in \mathbb{N}_2} \frac{b_n}{a_n} = 0.$$

Corollary 2: All the diagonal operators from $L_\varphi(a,r)$ to $L_\varphi(b,s)$, $0 < r,\ s < \infty$, are compact if and only if one of the following conditions is satisfied:

(a) $\overline{\lim_{n}} \frac{b_n}{a_n} < \frac{r}{-}$, or b) $\overline{\lim_{n}} \frac{a_n}{b_n} < \frac{s}{r}$, or c) there exists a partition $\mathbb{N} = \mathbb{N}_1 \setminus \mathbb{N}_2$ of the indices such that

$$\overline{\lim_{n \in \mathbb{N}_1}} \frac{b_n}{a_n} < \frac{r}{s} \text{ and } \overline{\lim_{n \in \mathbb{N}_2}} \frac{a_n}{b_n} < \frac{s}{r}.$$

Remark 3: An analysis of the preceding proofs shows that the existence of a non-compact diagonal operator from $L_\varphi(a,r)$ to $L_g(b,s)$ is related to the existence of a subsequence I of the indices with the property that the corresponding step-spaces $L_\varphi(a,r)_I$ and $L_g(b,s)_I$ are the same. It is therefore reasonable to ask whether a similar result holds when there exists a non-compact (not necessarily diagonal) operator between the spaces. This problem will be solved in a forthcoming paper. (Part II.)

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