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ON DISCONTINUITY OF THE SPECTRAL RADIUS IN BANACH ALGEBRAS

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Abstract: We give an example of a complex Banach algebra with two generators containing no non-zero quasinilpotents in which the spectral radius is discontinuous (even on lines).

Key words: Banach algebras, continuity of the spectral radius, quasinilpotents.

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Introduction: Banach algebras with uniformly continuous spectral radius have been recently characterized as those commutative modulo the radical, [1],[5],[10] . There remains the problem of algebraic characterization of Banach algebras on which the spectral radius is merely continuous [9] . In connection with the paper [7] the problem of investigation continuity properties of the spectral radius in Banach algebras without quasinilpotents has been raised by J. Zemánek in [9]. Here we give a negative answer to this question. So the closedness of the set of quasinilpotents is not sufficient for the continuity of the spectral radius (of course it is necessary).

Our construction is based on a modification of the example due to S. Kakutani ([6], p. 282) of discontinuity of the spectral radius. We obtain an example of two operators A, B on a Hilbert space such that the spectral radius is discontinuous on the line $A + \lambda B$, (λ real) (for related topics see [1],[8]). This gives in some sense the best possible example for we get a Banach algebra with two generators A, B with discontinuous spectral radius. On the other hand every Banach algebra with one generator is commutative and therefore its spectral radius is continuous even uniformly.

The second tool in our construction is an idea used by A.S. Nemirovskij [4] and J. Duncan and A.W. Tullo [3] in constructing the first example of a non-commutative Banach algebra without quasinilpotents (in fact, we do not know whether the spectral radius is continuous in this original example).

We start with the following lemma:

__Lemma__: There exists a sequence $\{\beta_j\}_{j\in N}$ of all rational numbers from the interval $(0,2)$ such that

$$(1) \qquad \beta_1^{2^{k-1}} \cdot \beta_2^{2^{k-2}} \cdot \ldots \cdot \beta_{k-1}^{2} \cdot \beta_k \geq 1$$

for every $k \in N$.

__Proof__. Let $\{r_i\}_{i\in N}$ and $\{s_i\}_{i\in N}$ be sequences of all rational numbers from the intervals $(0,1\rangle$ and $(1,2)$, respectively. Denote $R = \{r_i, i \in N\}$, $S = \{s_i, i \in N\}$. We construct the sequence $\{\beta_j\}_{j\in N}$ inductively as follows. Put $\beta_1 = s_1$. Let $\beta_1, \ldots, \beta_k$ be defined and $\{\beta_1, \ldots, \beta_k\} = \{r_1, \ldots, r_i\} \cup \{s_1, \ldots, s_j\}$. For simplicity put $f(k) =$

$= \beta_1^{2^{k-1}} \cdot \ldots \cdot \beta_k$. If $\beta_k = s_j$ and $f(k)^2 \cdot r_{i+1} \geq 1$ put $\beta_{k+1} = r_{i+1}$. Put $\beta_{k+1} = s_{j+1}$ otherwise.

It is clear from the construction that $f(k) \geq 1$ for every $k \in N$ so the condition (1) is satisfied. Denote $B = \{\beta_j, j \in N\}$. Clearly $S \subset B$ and so it suffices to prove $R \subset B$. Suppose the contrary. Let $R \cap B = \{r_1, \ldots, r_m\}$, $(m \geq 0)$. Let $r_m = \beta_k$. It follows from the construction that $\beta_{k+1} > 1$ so $f(k+1) = f(k)^2 \cdot \beta_{k+1} > 1$ and $f(\ell)^2 \cdot r_{m+1} < 1$ for every $\ell > k$. At the same time we have $f(\ell) = f(\ell - 1)^2 \cdot \beta_\ell > f(\ell - 1)^2 > \ldots > f(k+1)^{2^{\ell-k-1}}$.

So we have (for every $\ell > k$) $r_{m+1} < \dfrac{1}{f(\ell)^2} < \dfrac{1}{f(k+1)^{2^{\ell-k}}}$.

But the last term tends to 0 for $\ell \to \infty$, so $r_{m+1} \leq 0$, a contradiction. Hence the described sequence $\{\beta_j\}_{j \in N}$ really exhausts all rational numbers from $(0,2)$.

Example (discontinuity of the spectral radius on lines): Let H_1 be a separable Hilbert space with an orthonormal sequence of vectors $e_1, e_2, \ldots$. Let A_1 be a bounded linear operator on H_1 (shortly $A_1 \in B(H_1)$) defined by $A_1 e_k = e_{k+1}$, $k = 1,2,\ldots$ (A_1 is an unilateral shift). We define an operator $A_0 \in B(H_1)$ as a weighted shift with weights α_i, i.e. $A_0 e_k = \alpha_k \cdot e_{k+1}$, where $\alpha_k = \beta_j$ for $k = 2^{j-1}(2m+1)$, $m \in N$ (β_j are the numbers from the lemma above). Then obviously $A_0 - \lambda A_1$ is a nilpotent operator for every $\lambda \in (0,2)$ rational and $|A_0|_\sigma = \lim_{n \to \infty} |A_0^n|^{1/n} \geq \limsup_{k \to \infty} \left(\prod_{i=1}^{2^k-1} \alpha_i \right)^{1/(2^k-1)} = \limsup_{k \to \infty} f(k)^{1/(2^k-1)} \geq 1$. So $\lim_{\lambda \to 0} (A_0 - \lambda A_1) = A_0$ and $\lim_{\lambda \to 0} |A_0 - \lambda A_1|_\sigma \neq |A_0|_\sigma$ (in fact this limit does not exist as follows from the almost continuity theorem of Aupetit, see [2]).

<u>Theorem</u>: There exists a complex Banach algebra $\mathcal{B}$ with two generators t_0, t_1, without non-zero quasinilpotents such that the spectral radius is discontinuous on the line $t_0 + \lambda t_1$, λ real.

Remark: It is not clear whether the Banach algebra generated by the operators A_0, A_1 from the example above does not contain quasinilpotents. We avoid this difficulty by the following construction.

Construction: Let H_1, $\{e_m\}$, A_0, A_1 be as before. Let H_2 be the Hilbert space with the orthonormal basis $\{f_{i_1,i_2\ldots i_n;k}\}$, where $n \in N$, $1 \leq k \leq n$, $i_j \in \{0,1\}$ for $j = 1,\ldots,n$. Put $H = H_1 \oplus H_2$. Define operators T_0, $T_1 \in B(H)$ by the relations $T_j/H_1 = A_j$, $j = 0,1$,

$$T_j(f_{i_1,i_2\ldots i_n;k}) = \begin{cases} 0 & \text{if } i_k \neq j \\ \frac{1}{2}\cdot f_{i_1,i_2\ldots i_n;k+1} & \text{if } i_k = j \text{ and } k < n \\ \frac{1}{2}\cdot f_{i_1,i_2\ldots i_n,1} & \text{if } i_k = j \text{ and } k = n. \end{cases}$$

(For instance $T_0(f_{1,1,0,1;2}) = 0$, $T_1(f_{1,1,0,1;2}) = \frac{1}{2} f_{1,1,0,1;3}$.) Obviously $|(T_0 - \lambda T_1)/H_2| \leq \frac{1}{2}$ for every $|\lambda| \leq 1$.

Denote by $G_k \subset B(H)$, $k = 0,1,\ldots$ the smallest closed (in the norm topology) subspace of B(H) containing all operators of the form $T_{i_k} \cdot T_{i_{k-1}} \dots T_{i_1}$ (G_0 is the set of scalar multiples of identity). Denote $\mathcal{B} = \{\{S_0, S_1, \dots\}, S_i \in G_i, \sum_{i=0}^{\infty} |S_i| < \infty\}$. We shall introduce algebraic operations and a norm on $\mathcal{B}$ by

$$\{S_0, S_1, \dots\} + \{S_0', S_1', \dots\} = \{S_0 + S_0', S_1 + S_1', \dots\}$$

$$\lambda \cdot \{S_0, S_1, \dots\} = \{\lambda \cdot S_0, \lambda \cdot S_1, \dots\} \text{ for } \lambda \in C$$

$$\{S_0, S_1, \dots\} \cdot \{S_0', S_1', \dots\} = \{U_0, U_1, \dots\} \text{ where}$$

$$U_k = \sum_{i+j=k} S_i S_j' \text{ (convolution)}$$

and $\|\{S_0, S_1, \dots\}\| = \sum_{i=0}^{\infty} |S_i|$.

We shall denote by $\|\cdot\|$ the norm in $\mathcal{B}$ and by $|\cdot|$ the norm in B(H). In the same way $\|\cdot\|_\sigma$ and $|\cdot|_\sigma$ denotes the spectral radii in $\mathcal{B}$ and B(H), respectively. One can prove easily that $\mathcal{B}$ with the norm $\|\cdot\|$ is a Banach algebra with the unit $\{I, 0, 0, \dots\}$ and with generators $t_0 = \{0, T_0, 0, 0, \dots\}$ and $t_1 = \{0, T_1, 0, 0, \dots\}$.

We shall prove that $\mathcal{B}$ satisfies all the conditions required. Observe first that $\|\{0, 0, \dots, 0, S_k, 0, \dots\}\|_\sigma =$

$= \lim_{n \to \infty} \|\{0, \dots, 0, S_k^n, 0, \dots\}\|^{1/n} = \lim_{n \to \infty} |S_k^n|^{1/n} = |S_k|_\sigma$.

We have $\lim_{\lambda \to \infty} (t_0 - \lambda \cdot t_1) = t_0$. On the other hand

$\|t_0 - \lambda \cdot t_1\|_\sigma = \|\{0, T_0 - \lambda \cdot T_1, 0, \dots\}\|_\sigma = |T_0 - \lambda \cdot T_1|_\sigma \le$

$\le \frac{1}{2}$ for $\lambda \in (0,1)$, λ rational; and $\|t_0\|_\sigma =$

$= \|\{0, T_0, 0, \dots\}\|_\sigma = |T_0|_\sigma \ge |T_0/H_1|_\sigma \ge 1$. So the spectral radius is discontinuous on the line $t_0 + \lambda \cdot t_1$, λ real.

It remains to prove that $\mathcal{B}$ does not contain non-zero quasinilpotents. Suppose the contrary. Let $\{S_0, S_1, \dots\}$ be a non-zero quasinilpotent in $\mathcal{B}$. Let k be the smallest index such that $S_k \neq 0$. Then for every $n \in N$ it holds $\| \{S_0, S_1, \dots\}^n \| \geq | S_k^n |$, therefore $0 = \| \{S_0, S_1, \dots\} \|_6 \geq$ $\geq | S_k |_6$ and $| S_k |_6 = 0$.

As $S_k \in G_k$, it can be written in the form $S_k = \lim_{r \to \infty} S_k^{(r)}$ (in the norm of B(H)), $S_k^{(r)} = \sum_{i_1 \dots i_k} \lambda^{(r)}_{i_1, \dots i_k} \cdot T_{i_k} \dots T_{i_1}$ (finite sum), $i_1, \dots, i_k \in \{0,1\}$. As $| S_k |_6 = 0$ and the vectors $f_{i_1 \dots i_k, i}$ are eigenvectors of S_k it must be $0 =$

$= S_k(f_{i_1 \dots i_k, 1}) = \lim_{r \to \infty} S_k^{(r)}(f_{i_1 \dots i_k, 1}) =$

$= \lim_{r \to \infty} \sum_{j_1 \dots j_k} \lambda^{(r)}_{j_1 \dots j_k} T_{j_k} \dots T_{j_1}(f_{i_1 \dots i_k, 1}) =$

$= \lim_{r \to \infty} \lambda^{(r)}_{i_1 \dots i_k} \left(\frac{1}{2}\right)^k \cdot f_{i_1 \dots i_k, 1}$. Hence

(2) $\lim_{r \to \infty} \lambda^{(r)}_{i_1 \dots i_k} = 0$ for every $i_1, \dots, i_k \in \{0,1\}$.

Supposing (2) is satisfied we shall enumerate

$S_k(f_{i_1 \dots i_t, s}) = \lim_{r \to \infty} \sum_{j_1 \dots j_k} \lambda^{(r)}_{j_1 \dots j_k} \cdot T_{j_k} \dots T_{j_1}(f_{i_1 \dots i_t, s}) =$

$= \lim_{r \to \infty} \left(\frac{1}{2}\right)^k \cdot \lambda^{(r)}_{i_1, i_{s+1}, \dots i_{s+k-1}} \cdot f_{i_1 \dots i_t, s+k} = 0$

(addition is to be taken mod t) and $S_k(e_m) = \lim_{r \to \infty} S_k^{(r)}(e_m) =$

$= \lim_{r \to \infty} \sum_{i_1 \dots i_k} \lambda^{(r)}_{i_1, \dots i_k} \cdot T_{i_k} \dots T_{i_1}(e_m) =$

$= \sum_{i_1 \dots i_k} \lim \lambda^{(r)}_{i_1, \dots i_k} \cdot T_{i_k} \dots T_{i_1}(e_m) = 0$ according to (2). So we have $S_k = 0$, a contradiction.

This finishes the proof.

R e f e r e n c e s

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