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Gerald W. Johnson; Loren V. Petersen

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A NOTE ON BANACH SPACES WITHOUT THE APPROXIMATION PROPERTY

G.W. JOHNSON and Loren V. PETERSEN, Lincoln

Abstract: We show that a certain class of Banach sequence spaces has closed subspaces without the approximation property. Certain Lorentz sequence spaces provide particularly interesting examples.

Key words: Banach space, approximation property, Lorentz sequence space, Banach sequence space.

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1. Introduction. Grothendieck [4] asked whether every Banach space has the approximation property (a.p.). Enflo [3] showed that there is a closed subspace of the Banach space c_0 which fails to have a.p. Since then further papers have appeared giving examples of Banach spaces without a.p. One of the most notable is the paper by Davie [2]. He shows that not only c_0 but also the ℓ_p spaces $(2< p < \infty)$ have closed subspaces without a.p. The purpose of this note is to show that some simple modifications of Davie's elegant construction can be used to show that a certain class of Banach sequence spaces all have closed subspaces which fail to have a.p. We will mention a variety of spaces to which this result can be applied but we will focus attention on a

class of Lorentz sequence spaces.

Let $w = (w_n)$ be a nonincreasing sequence of positive numbers with $\lim w_n = 0$ and $\sum w_n = \infty$. For $1 \leq p < \infty$, let $d(w,p)$ denote the space of all complex sequences $a = (a_n)$ for which $\| a \| = \sup \left(\sum | a_{\pi(n)} |^p w_n \right)^{1/p} < \infty$, the supremum being taken over all permutations π of the positive integers. The space $d(w,p)$ endowed with the norm $\| \cdot \|$ is a Banach space called a Lorentz sequence space. For information on Lorentz sequence spaces see [1, 5].

The spaces $d(w,p)$ resemble in certain respects the ℓ_p spaces; in particular, they share with the ℓ_p-spaces the property that each of their infinite dimensional closed subspaces contains a closed subspace isomorphic to ℓ_p [5; p. 30]. In light of this and since the question as to whether $\ell_p (1 \leq p < 2)$ has a closed subspace without a.p. is unsettled and apparently difficult, one might guess that it would be hard to decide whether or not $d(w,p)$ $(1 \leq p < 2)$ has a closed subspace without a.p. However, we will see that if the weighting sequence (w_n) satisfies an appropriate growth condition, then it is an easy consequence of our result that $d(w,p)$ has a closed subspace $E(w,p)$ without a.p. There is another interesting aspect of the examples $E(w,p)$ $(1 \leq p \leq 2)$. When examples of Banach spaces without a.p. have been constructed, no consideration seems to have been given as to whether or not these were isomorphic to the examples already constructed by Davie. It will be easy to see that the spaces $E(w,p)$ $(1 \leq p \leq 2)$ are not isomorphic to any of the Davie spaces.

2. Davie's Construction for Certain Banach Sequence Spaces. Let X be a countably infinite set. Let M^+ denote the set of nonnegative functions on X with values in the extended reals. A map ρ on M^+ to the extended reals is a function norm if ρ satisfies the following for all f, g in M^+: (i) $\rho(f) \geq 0$ and $\rho(f) = 0$ if and only if $f = 0$, (ii) $\rho(af) = a\rho(f)$ for all constants $a \geq 0$, (iii) $\rho(f + g) \leq \rho(f) + \rho(g)$, (iv) $f \leq g$ implies $\rho(f) \leq \rho(g)$. We extend the definition of ρ to M, the set of all complex-valued functions on X by $\rho(f) = \rho(|f|)$. Let ℓ_ρ denote the set of all f in M satisfying $\rho(f) < \infty$. It is well known [7; p. 444] that (ℓ_ρ, ρ) is a Banach space if and only if ρ has the property that for every sequence (f_n) of nonnegative functions in ℓ_ρ satisfying $\sum \rho(f_n) < \infty$, we have $\rho(\sum f_n) \leq \sum \rho(f_n)$. We will refer to such a Banach space (ℓ_ρ, ρ) as a Banach sequence space. These spaces are special cases of the "Banach function spaces" studied extensively by Luxemburg and Zaanen and others (see [7; Chapter 15]).

We now modify Davie's construction to fit the setting of a Banach sequence space. The probabilistic steps in Davie's proof need no modification and we will refer the reader to [2] for these steps.

Let (ℓ_ρ, ρ) be a Banach sequence space. Let $(G_k)_{k=0}^{\infty}$ be a sequence of disjoint subsets of X with the cardinality of G_k being $3 \cdot 2^k$. Define $H_k \equiv G_{k-1} \cup G_k \cup G_{k+1}$ for $k \geq 0$ where we take $G_{-1} = \phi$. Let $G \equiv \bigcup_{k=0}^{\infty} G_k$.

Lemma. If there is $\eta > 0$ such that $\eta < \rho(\chi_{\{g\}}) < \infty$ for all g in G, then $\| u \|_\infty < \eta^{-1} \rho(u)$ for all u in ℓ_ρ supported by G.

Proof. Take u in ℓ_ρ supported by G. Let g belong to G. Then $\rho(u) \geq \rho(|u(g)| \chi_{\{g\}}) = |u(g)| \rho(\chi_{\{g\}}) \geq |u(g)| \eta$.

Theorem 1. Suppose that for $\eta > 0$, $\eta < \rho(\chi_{\{g\}}) < \infty$ for all g in G. Further suppose there is $2 < r \leq \infty$ and a constant C such that

(1) $$\rho(\chi_{H_k}) \leq C \| \chi_{H_k} \|_r$$

for all $k \geq 0$. Then there is a closed subspace of ℓ_ρ without a.p.

Remark. Condition (1) is the key. Even for rather complicated norms ρ it seems to be easy to check as it involves only the easily dealt with functions χ_{H_k}.

Proof. Regard the G_k's as cyclic groups. It is shown in [2] via a probabilistic argument that the $3 \cdot 2^k$ characters on G_k can be partitioned into $\sigma^k_1, \dots, \sigma^k_{2^k}$ and $\tau^k_1, \dots, \tau^k_{2^{k+1}}$ so that

(2) $$\left| 2 \sum_{j=1}^{2^k} \sigma^k_j(g) - \sum_{j=1}^{2^{k+1}} \tau^k_j(g) \right| \leq A_2 (k+1)^{1/2} 2^{k/2}$$

for some constant A_2 and all g in G_k. Let $\varepsilon^k_j = \pm 1$ for $k \geq 0$, $1 \leq j \leq 2^k$. (For now the choice of ε^k_j will not matter. Later we will need a choice that will yield inequality (11) below.)

Define functions e_j^k $(k \geq 0,\ 1 \leq j \leq 2^k)$ by

$$(3) \qquad e_j^k(g) = \begin{cases} \tau_j^{k-1}(g), & g \text{ in } G_{k-1} (k \geq 1) \\ \varepsilon_j^k \sigma_j^k(g), & g \text{ in } G_k \\ 0 \quad , & \text{otherwise.} \end{cases}$$

Then $(e_j^k) \subset \ell_\rho$ since each e_j^k has finite support in G and $\rho(\chi_{\{g\}}) < \infty$ for all g in G. Let E be the closed span of (e_j^k).

Define the complex linear functionals α_j^k $(k \geq 0,\ 1 \leq j \leq 2^k)$ by

$$(4) \qquad \alpha_j^k(f) = 3^{-1} \cdot 2^{-k} \sum_{g \text{ in } G_k} e_j^k \sigma_j^k(g^{-1}) f(g)$$

for all f in E. Then $(\alpha_j^k) \subset E^*$, the dual of E. We see this by taking (f_n) and f in E such that $\rho(f_n - f) \longrightarrow 0$. Then $|\alpha_j^k(f_n - f)| \leq \max_{g \text{ in } G_k} |f_n(g) - f(g)| \longrightarrow 0$ as $n \longrightarrow \infty$ since $|f_n(g) - f(g)| \cdot \rho(\chi_{\{g\}}) \leq \rho(f_n - f)$ for all g in G. Similarly one can see that the right side of (5) below is in E^*. Further $\alpha_j^k(e_i^\ell) = \delta_{ij} \cdot \delta_{k\ell}$ in (4) and (4) agrees with the right side of (5) on $(e_i^\ell : \ell \geq 0,\ 1 \leq i \leq 2^\ell)$. Therefore since both (4) and the right side of (5) are continuous and linear, we have for $k \geq 1$ and f in E

$$(5) \qquad \alpha_j^k(f) = 3^{-1} \cdot 2^{1-k} \sum_{g \text{ in } G_{k-1}} \tau_j^{k-1}(g^{-1}) f(g).$$

Define linear functionals β^k $(k \geq 0)$ on the bounded linear operators from E to E by

$$\beta^k(T) = 2^{-k} \sum_{j=1}^{2^k} \alpha_j^k(Te_j^k).$$

Since $\alpha_j^k(e_j^k) = 1$, we have $\beta^k(I) = 1$ where I is the identity operator. Using (4) we obtain

(6) $$\beta^k(T) = 3^{-1}.4^{-k} \sum_{g \text{ in } G_k} T\Big(\sum_{j=1}^{2^k} \varepsilon_j^k \sigma_j^k(g^{-1})e_j^k\Big)(g),$$

and using (5) with k + 1 instead of k,

(7) $$\beta^{k+1}(T) = 6^{-1}.4^{-k} \sum_{g \text{ in } G_k} T\Big(\sum_{j=1}^{2^{k+1}} \tau_j^k(g^{-1})e_j^{k+1}\Big)(g).$$

(6) and (7) yield

(8) $$\beta^{k+1}(T) - \beta^k(T) = 3^{-1}.2^{-k} \sum_{g \text{ in } G_k} T(\phi_g^k)(g),$$

where

$$\phi_g^k = 2^{-k-1} \sum_{j=1}^{2^{k+1}} \tau_j^k(g^{-1})e_j^{k+1} - 2^{-k} \sum_{j=1}^{2^k} \varepsilon_j^k \sigma_j^k(g^{-1})e_j^k.$$

Then the support of ϕ_g^k is in H_k and ϕ_g^k is in E for all g in G_k. From (8) and the Lemma we see that

(9) $$|\beta^{k+1}(T) - \beta^k(T)| \leq \sup_{g \text{ in } G_k} \| T\phi_g^k \|_\infty \leq$$

$$\leq \eta^{-1} \cdot \sup_{g \text{ in } G_k} \wp(T\phi_g^k).$$

Davie shows via a probabilistic argument that (ε_j^k) can be chosen in (3) above so that

(10) $$|\phi_g^k(h)| \leq A_3(k+1)^{1/2}2^{-k/2}$$

for h in H_k and g in G_k. From our hypothesis there is

$2 < r \leq \infty$ and a constant A_4 such that

(11) $$\rho(\phi_g^k) \leq A_4(k+1)^{1/2} 2^{-k/2} \| \chi_{H_k} \|_r .$$

Now define $K \equiv \{0\} \cup \{e_1^0\} \cup \{(k+1)^2 \phi_g^k : k \geq 0,\ g \text{ in } G_k\} \subset E$. By (11), K is compact in ℓ_ρ and by (9),

$$|\beta^{k+1}(T) - \beta^k(T)| \leq \eta^{-1}(k+1)^{-2} \sup_{x \text{ in } K} \rho(Tx)$$

for all bounded linear operators T from E to E. Further $|\beta^0(T)| \leq \eta^{-1} \sup_{x \text{ in } K} \rho(Tx)$. Hence, for all bounded linear operators T from E to E, $\beta(T) = \lim_k \beta_k(T)$ exists and satisfies

(12) $$|\beta(T) \leq 3\eta^{-1} \sup_{x \text{ in } K} \rho(Tx).$$

Then since $\beta^k(I) = 1$ for all k,

(13) $$\beta(I) = 1.$$

Now $S_n \longrightarrow S$ uniformly on K in ℓ_ρ implies by (12) that $\lim |\beta(S_n) - \beta(S)| = 0$. Now if $Tx = \psi(x)e_i^k$, ψ in E^*, then $\beta^\ell(T) = 0$ for $\ell > k$ and so $\beta(T) = 0$. By the linearity of β, $\beta(T) = 0$ when $Tx = \psi(x)z$ for ψ in E^* and z in span(e_i^k). Now for $Tx = \psi(x)z$ with ψ in E^* and z in E take $(z_n) \subset$ span(e_i^k) such that $\rho(z_n - z) \longrightarrow 0$. Let $T_n x = \psi(x)z_n$. Then

$\| T_n - T \| = \sup_{\rho(x) \leq 1} \rho(T_n x - Tx) \leq (\sup_{\rho(x) \leq 1} |\psi(x)|)\rho(z_n - z) \to \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $\beta(T_n) \longrightarrow \beta(T)$, so $\beta(T) = 0$. By the linearity of β, $\beta(T) = 0$ for all finite rank operators T from E to E. It now follows from (13) and the sentence following (13) that there cannot be a sequence of finite rank operators (T_n) from E to E such that $T_n \longrightarrow I$ uniformly on K. Therefore E fails to have a.p.

3. Examples. Since the above is a simple modification of Davie's argument it is not surprising that Davie's result should follow easily from it. Take (G_k) to be a sequence of disjoint subsets of the positive integers of the proper cardinality. For $2 < p < \infty$, let φ be the ℓ_p-norm and take $r = p$ and $C = 1$. Theorem 1 implies that there is a closed subspace of ℓ_p, call it E_p, without a.p. Alternately let φ be the ℓ_∞ norm except take $\varphi = \infty$ for any sequence which fails to converge to 0. Then ℓ_φ is a Banach sequence space [7; p. 449, Exercise 65.2]; in fact $(\ell_\varphi, \varphi) = (c_0, \|\cdot\|_\infty)$. In this case taking $r = \infty$ and $C = 1$ and applying the theorem, one gets a closed subspace E_∞ of c_0 without a.p.

Remark. The subspace E_p is not uniquely determined by p. In particular any choice of the (ε_j^k) for which inequality (10) holds produces a closed subspace, $E_p(\varepsilon_j^k)$, without a.p. One may think of choosing the sequence (ε_j^k) of ± 1's at random. If this is done, one may ask for the probability of obtaining a subspace without a.p. It is an easy consequence of the Borel-Cantelli Lemma [6; p. 70] that the probability is one.

Theorem 2. Take (G_k) to be a sequence of disjoint subsets of the positive integers of proper cardinality and let $1 \leq p < \infty$. Let $d(w,p)$ be a Lorentz sequence space consisting of sequences supported by G. Suppose there exists $r > 2$ and a number M such that

$$(14) \qquad \left(\sum_{n=1}^{k} w_n\right)^{1/p} \leq Mk^{1/r}, \quad k = 1,2,\ldots .$$

Then d(w,p) has a closed subspace E(w,p) without a.p. In particular, $d((\frac{1}{n}),p)$ has a closed subspace $E((\frac{1}{n}),p)$ without a.p. for all p, $1 \leq p < \infty$.

Proof. Let ρ be the Lorentz norm. For g in G, $\rho(\chi_{\{g\}}) = w_1^{1/p} > 0$. Also by (14)
$\rho(\chi_{H_k}) \leq M(3 \cdot 2^{k-1} + 3 \cdot 2^k + 3 \cdot 2^{k+1})^{1/r} = M \, \|\chi_{H_k}\|_r$. Hence d(w,p) has a closed subspace without a.p.

The fact that $d((\frac{1}{n}),p)$ has a closed subspace without a.p. for all p ($1 \leq p < \infty$) follows from the fact that
$\lim_{k \to \infty} \left(\sum_{n=1}^{k} \frac{1}{n} - \ln k\right) = \gamma$, Euler's constant.

Given the Davie space E_p ($2 < p \leq \infty$) it is easy to generate further spaces without a.p.; indeed $E_p \oplus B$ is such a space for any Banach space B. It is natural to ask if our spaces are distinct from these.

Theorem 3. Let $1 \leq p \leq 2$ and let w satisfy the hypotheses of Theorem 2. Then d(w,p) is nonisomorphic to $E_r \oplus B$ for any choice of r ($2 < r \leq \infty$) and any choice of a Banach space B.

Proof. Fix an appropriate d(w,p). Let r and B be given. Suppose d(w,p) were isomorphic to $E_r \oplus B$. E_r, being a closed infinite-dimensional subspace of ℓ_r, has a closed subspace isomorphic to ℓ_r [5; p. 30]. But then d(w,p) would have a closed subspace F isomorphic to ℓ_r. But F, being a closed subspace of d(w,p), would have a further closed subspace isomorphic to ℓ_p. Thus $\ell_r \cong F$ has ℓ_p as a closed subspace where $r \neq p$. But it is well known that this cannot happen [5; p. 31].

We finish by briefly mentioning some further examples that can be obtained from Theorem 1. Let $(p_j)_{j=0}^{\infty}$ be given so that $1 \leq p_0 < \infty$ and $2 < r \leq p_j < \infty$ for $j = 1,2,\dots$. The Banach space $A = (\sum_{j=1}^{\infty} \oplus \ell_{p_j})_{p_0}$ can be regarded as a Banach sequence space where X is taken as the set of pairs of positive integers. Subspaces E of A without a.p. can be constructed in a variety of ways. Unlike the earlier examples the choice of the sequence (G_k) affects the nature of the subspace. As one example it is possible to take the G_k's so that E contains closed subspaces isomorphic to each of ℓ_{p_j}, $j = 0,1,2,\dots$. Such an E is clearly not isomorphic to any of the Davie spaces E_p nor to any of the spaces $E(w,p)$ from Theorem 2. Theorem 1 may also be applied to certain of the Orlicz sequence spaces.

R e f e r e n c e s

[1] Z. ALTSHULER, P.G. CASAZZA and BOR-LUH LIN: On symmetric basic sequences in Lorentz sequence spaces, Israel J. of Math. 15(1973), 140-155.

[2] A.M. DAVIE: The approximation problem for Banach spaces, Bull. London Math. Soc. 5(1973), 261-266.

[3] P. ENFLO: A counterexample to the approximation problem, Acta Math. 130(1973), 309-317.

[4] A. GROTHENDIECK: Produits tensorieles topologiques et espaces nucléaires, Memoirs Amer. Math. Soc., No. 16, 1955.

[5] J. LINDENSTRAUSS and L. TZAFRIRI: Classical Banach spaces, Lect. Notes in Math., vol. 338, Springer-Verlag, New York, 1973.

[6] H.G. TUCKER: A Graduate Course in Probability, Academic Press, New York, 1967.

[7] A.C. ZAANEN: Integration, Second edition, North-Holland, Amsterdam, 1967.

Department of Mathematics
University of Nebraska
Lincoln, Nebraska 68588
U.S.A.

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