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A NOTE ON DIRECT-PRODUCT DECOMPOSITIONS

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Abstract: An example of a covariant functor F such that the category S(F) does not admit the algebraic recognition of products (see [4]) is constructed.

Key words: Algebraic recognition of products, n-ary operation, set-functor.

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1. G.M. Kelly and A. Pultr have defined a condition of the algebraic recognition of products in categories (ARP, see [4]). Roughly speaking, in categories admitting ARP an object A can be non-trivially decomposed into a product of objects $A_1,\dots,A_n$ iff there exists a non-trivial n-ary operation e on A satisfying certain conditions. (For n = 2, and $A^2 \underset{p_1}{\overset{p_0}{\rightrightarrows}} A$ projections, the conditions mentioned are the following: $e\Delta = 1$, $e(e\times e) = e(p_0\times p_1)$ where Δ is a diagonal map.)

In [4], large classes of categories admitting ARP are presented. An interesting question is to study this problem in categories $\hat{F}$ defined as follows (cf. [2],[3]): consider a functor $F: \mathcal{A} \longrightarrow \mathrm{Set}$, define $\hat{F}$ as a category whose objects are all the pairs (A,a) where $A \in \mathrm{obj}\,\mathcal{A}$, $a \subset FA$, and whose morphisms $(A,a) \longrightarrow (B,b)$ are maps $f: A \longrightarrow B$ satisfying

$f_* a \subset b$. (The notation f_* (f^* resp.) is used for the direct- (inverse- resp.) image function.)

2. Thus, for $\mathcal{A}$ = Set (Set^{op} resp.), $\hat{F}$ ($\hat{F}^{op}$ resp.) coincides with the S(F) from (e.g.) [2] and [3].

While, by [4], S(F) with a contravariant F always admits ARP, for the covariant case only a class of the F (including the basic "constructive" set functors and closed under basic operations) with S(F) admitting ARP is given.

We are going to present a covariant functor F: Set $\longrightarrow$ Set such that in S(F) ARP is not admitted. For $\mathcal{A}$ a category admitting ARP (e.g. $\mathcal{A}$ = Set), F admits ARP for n = 2 iff the following condition holds:

$$(\underline{1}_{A_0,A_1}) \quad \text{Let } (A_0\times A_1)^2 \underset{p_1}{\overset{p_0}{\rightrightarrows}} A_0\times A_1, \quad A_0\times A_1 \begin{matrix} \overset{\pi_0}{\nearrow} A_0 \\ \underset{\pi_1}{\searrow} A_1 \end{matrix}$$

be product diagrams in $\mathcal{A}$ then for every $r \subset F(A_0\times A_1)$:

$$F(\pi_0\times \pi_1)_* (F(p_0)^* (r)\cap F(p_1)^* (r))\subset r \text{ implies}$$

$$F(\pi_0)^* F(\pi_0)_* (r)\cap F(\pi_1)^* F(\pi_1)_* (r)\subset r.$$

For a set X put

$$FX = \{ Y\subset X \mid \mathrm{card}\ Y = 2 \}\cup \{ 0_X \}$$

and for a mapping $f: X \longrightarrow X'$ define $F(f)$ by putting

$$F(f)(Y) = f_* (Y) \text{ if card } f_* (Y) = 2,$$

$$F(f)(Y) = 0_{X'} \text{ if card } f_* (Y) = 1,$$

$$F(f)(0_X) = 0_{X'} .$$

(Thus, F is a factorfunctor of Hom (2,-) where all the constant maps of Hom (2,X) are factorized to a point of FX.)

<u>Proposition</u>. The $\hat{F}$ just defined does not admit ARP.

Proof. Put $A_0 = A_1 = 2$, and consider the subset $r =$ $= \{\{(0,0),(0,1)\}\} \subset F(2\times 2)$. Then

$F(p_0)^*(r) = \{\{(0,0,a,b),(0,1,c,d)\} \mid a,b,c,d \in \{0,1\}\}$,

$F(p_1)^*(r) = \{\{(x,y,0,0),(u,v,0,1)\} \mid x,y,u,v \in \{0,1\}\}$,

$F(p_0)^*(r) \cap F(p_1)^*(r) = \{\{(0,0,0,0),(0,1,0,1)\}, \{(0,0,0,1),$
$(0,1,0,0)\}\}$,

$F(\pi_0 \times \pi_1)_*(F(p_0)^*(r) \cap F(p_1)^*(r)) = r$.

On the other hand, $F(\pi_0)_*(r) = \{0_2\}$,

$F(\pi_0)^* F(\pi_0)_*(r) = \{\{(0,0),(0,1)\}, \{(1,0),(1,1)\}, 0_{2\times 2}\}$,

$F(\pi_1)_*(r) = \{\{0,1\}\}$,

$F(\pi_1)^* F(\pi_1)_*(r) = \{\{(0,0),(0,1)\}, \{(1,0),(1,1)\}, \{(0,0),$
$(1,1)\}, \{(1,0),(0,1)\}\}$,

$F(\pi_0)^* F(\pi_0)_*(r) \cap F(\pi_1)^* F(\pi_1)_*(r) = \{\{(0,0),(0,1)\},$
$\{(1,0),(1,1)\}\} \not\subset r$.

3. Proving ARP property for the S(F) with concrete F's (representable functors, power-set functors, products and sums of these etc.) one usually encounters the situation with F satisfying the following formally stronger condition:

$(\underline{2}_{A_0,A_1})$ for any $u,v,w \in F(A_0 \times A_1)$ such that $F(\pi_0)(u) =$ $= F(\pi_0)(v)$, $F(\pi_1)(u) = F(\pi_1)(w)$ there exists a $z \in F((A_0 \times A_1)^2)$ such that $F(p_0)(z) = v$, $F(p_1)(z) = w$, $F(\pi_0 \times \pi_1)(z) = u$.

It is still an open problem whether this is equivalent with ARP, i.e., whether

$$\forall A_0,A_1\ (\underline{1}_{A_0,A_1}) \Longrightarrow \forall A_0,A_1\ (\underline{2}_{A_0,A_1}).$$

We will conclude this note by showing at least that the implication

$$(\underline{1}_{A_o,A_1}) \Longrightarrow (\underline{2}_{A_o,A_1})$$

with particular A_o, A_1 (namely already with $A_o = A_1 = 2$) does not hold.

For a set X put

$GX = \{(Y,i) \mid Y \subset X,\ \mathrm{card}\ Y = 4,\ i \in \{0,1\}\} \cup \{0_X\}$

and for a mapping $f: X \longrightarrow X'$ define $G(f): G(X) \longrightarrow G(X')$ by putting

$G(f)(Y,i) = (f_*(Y),i)$ if $\mathrm{card}\ f_*(Y) = 4$,

$G(f)(Y,i) = 0_{X'}$ if $\mathrm{card}\ f_*(Y) < 4$,

$G(f)(0_X) = 0_{X'}$.

<u>Proposition</u>. $(\underline{1}_{2,2})$ holds while $(\underline{2}_{2,2})$ does not.

<u>Proof</u>. (a) Let $r = \emptyset$. Then $G(\pi_o)^* G(\pi_o)_* (r) \cap \cap G(\pi_1)^* G(\pi_1)_* (r) = \emptyset$.

(b) Let $\emptyset \neq r \subset G(2\times 2) \setminus \{0_{2\times 2}\}$. We can suppose, without loss of generality, that $(2\times 2,0) \in r$. Then $y = = (\{(0,0,0,0),(0,1,1,0),(1,0,0,1),(1,1,1,1)\},0) \in \in G(p_o)^* (r) \cap G(p_1)^* (r)$ while $G(\pi_o \times \pi_1)(y) = 0_{2\times 2} \notin r$.

(c) Let $\emptyset \neq r \subset G(2\times 2) \setminus \{(2\times 2,i)\}$, $i = 0$ or $i = 1$. According to (b) we can suppose that $0_{2\times 2} \in r$. Then $t = = (\{(0,0,0,0),(0,0,1,1),(1,1,0,0),(1,1,1,1)\},i) \in \in G(p_o)^* (r) \cap G(p_1)^* (r)$ but $G(\pi_o \times \pi_1)(t) = (2\times 2,i) \notin r$.

(d) Let $r = G(2\times 2)$. Then $G(\pi_o)^* G(\pi_o)_* (r) \cap \cap G(\pi_1)^* G(\pi_1)_* (r) = r$.

According to (a) - (d) $(\underline{1}_{2,2})$ holds.

(e) Put $u = v = (2\times 2,0)$, $w = (2\times 2,1)$. Then $G(\pi_o)(u) = = G(\pi_1)(u) = G(\pi_o)(v) = G(\pi_1)(w) = 0_2$ but

$G(p_0)^* (v) \cap G(p_1)^* (w) = \emptyset$ and therefore $(\underset{=}{2}_{2,2})$ does not hold.

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