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SIMULATIONS OF PAWLAK MACHINES AS FUZZY MORPHISMS OF PARTIAL ALGEBRAS

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Abstract: Simulations of Pawlak machines are shown to be fuzzy morphisms (in the sense of Arbib and Manes) over certain category of unary partial algebras.

Key words: Pawlak machine, simulation, fuzzy theory, coreflective subcategory.

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Introduction: When studying automata of the non-deterministic type, M.A. Arbib and E.G. Manes introduced in [3] the notion of fuzzy theory over the category K. In fact, it is a category with the same class of objects as K in which morphisms, i.e. fuzzy morphisms, from a to b are morphisms of the category K from a to the object T(b); T(b) being interpreted as "the cloud of fuzzy states over the object of pure states" (cf. [3]). Arbib and Manes show that when certain natural requirements for the composition of fuzzy morphisms and for the relation between b and T(b) are fulfilled the category with fuzzy morphisms is a Kleisli category of suitable monad (cf. [3]).
Essentially, an analogous approach is that of H. Ehrig and col. in [6].

In [3],[4] are found many examples of fuzzy theories, especially of fuzzy theories over the category of sets, which among other things make possible the study of the non-deterministic sequential machines, stochastic sequential machines, semiring automata etc.

In this paper we give another example of fuzzy theory - we prove that simulations of Pawlak machines are fuzzy morphisms over certain category of unary partial algebras.

I. As mentioned in [3], fuzzy theories are closely related to coreflective subcategories with the same class of objects: let K be a coreflective subcategory of H, obj K = = obj H, P: H $\longrightarrow$ K be a coreflector and J: K $\longrightarrow$ H the inclusion functor. Put $T = P \circ J$. Let e be the natural transformation from 1_K to T, given by the adjoint situation, let $\odot$ be the composition in the category H. Then $(T, e, \odot)$ will be a fuzzy theory over K in the sense of [3] and H isomorphic to the category with fuzzy morphisms.

In fact, we prove that the category of all partial algebras and all their homomorphisms of a certain type (cf. below) is a coreflective subcategory of the category of all Pawlak machines and all their simulations. These two categories have the same class of objects, as will become clear.

II. Let us recall that a Pawlak machine is an ordered pair (A,f) in which f is a partial mapping from A to A, i.e. (A,f) is a partial algebra with one unary operation (cf. [1] and [2]). In accordance with [5] a mapping $\alpha : A \longrightarrow B$ is said to be a simulation of (A,f) in (B,g) if two following conditions are fulfilled:

$$(\forall a \in A)(a \in D(f) \text{ iff } \alpha(a) \in D(g))$$

$$(\forall a \in D(f))(\exists k_a \geq 1)(\alpha f(a)) = g^{k_a}(\alpha(a))),$$

where $D(f)$ and $D(g)$ are domains of f and g respectively, k_a is an integer and g^{k_a} denotes the k_a-th iteration of g. (As far as we know this notion of a simulation was first conceived by Z. Pawlak, although to our knowledge he has not published as yet any paper in which this notion appears.)

An usual homomorphism of partial algebras $\alpha : (A,f) \longrightarrow (B,g)$ which is also a simulation (and in that case the minimal $k_a = 1$ for all $a \in D(f)$) is called s-homomorphism.

The aim of this note is to prove the following proposition.

<u>Proposition</u>: The category of all Pawlak machines and all their s-homomorphisms is a coreflective subcategory of the category of all Pawlak machines and all their simulations.

III. <u>The proof of the proposition</u>

<u>Definition 1</u>: A Pawlak machine $(J,+)$ is said to be additive if J is either the set of all non-negative integers or $\{0,\dots,k\}$ (k is non-negative) and if the partial mapping $+ : J \longrightarrow J$ is defined in the following way:

(1) $i \in D(+)$ iff $i + 1 \in J$

(2) $(\forall i \in D(+))\ (+(i) = i + 1)$.

<u>Definition 2</u>: A simulation (resp. s-homomorphism) of and additive Pawlak machine $(J,+)$ in (A,f) is said to be a path (resp. s-path) in (A,f).

<u>Lemma 1</u>: Let $\iota : (J,+) \longrightarrow (A,f)$ be an s-path. Then

(i) $(\forall k \in J)(\iota k) = f^k(\iota(0)))$

(ii) $(\forall k > 0)(k \in J$ iff $\iota(0) \in D(f^k)$.

Proof: (i) we prove easily by induction.

(ii) Let us have $k > 0$. By definition 1 $k \in J$ iff $k - 1 \in D(+)$. ι is a simulation, thus $k - 1 \in D(+)$ iff $\iota(k - 1) \in D(f)$. From (i) it follows that $k - 1 \in D(+)$ iff $f^{k-1}(\iota(0)) \in D(f)$ iff $\iota(0) \in D(f^k)$. Hence, $k \in J$ iff $\iota(0) \in D(f^k)$.

Lemma 2: Let (A,f) be a Pawlak machine, let $a \in A$. Then there is only one s-path ι_a in (A,f) such that $\iota_a(0) = a$.

Proof: 1. Let us have $a \in A$. Define J_a: $i \in J_a$ iff $a \in D(f^i)$ $(f^0 = 1)$, and $+ : J_a \longrightarrow J_a$: $i \in D(+)$ iff $i + 1 \in J_a$, $+(i) = i + 1$. Define $\iota(i) = f^i(a)$. Obviously, ι_a is an s-path.

2. Now, let $\iota : J \longrightarrow A$ be an s-path, let $\iota(0) = a$. By Lemma 1 $k \in J$ iff $\iota(0) \in D(f)(i(0) = a \in D(f^0))$ and $\iota k = f^k(\iota(0))$. Hence, $J_a = J$ and $\iota_a = \iota$.

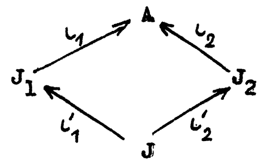
Construction. Let (A,f) be a Pawlak machine. Define $P((A,f)) = \{\langle i, \iota \rangle \mid \iota$ is a path in (A,f), $i \in D(\iota)\}$,

f': $P((A,f)) \longrightarrow P((A,f))$: 1. $\langle i, \iota \rangle \in D(f')$ iff $\iota(i) \in D(f)$

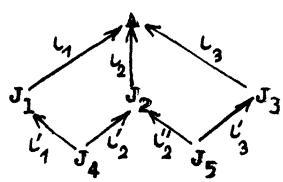
2. $f'(\langle i, \iota \rangle) = \langle i + 1, \iota \rangle$.

Then $(P((A,f)),f')$ is a Pawlak machine. Now we define the binary relation R on $P((A,f))$:

$\langle i, \iota_1 \rangle R \langle j, \iota_2 \rangle$ if there are J and a pair of s-paths ι'_1, ι'_2 such that $\iota'_1(0) = i$, $\iota'_2(0) = j$ and the diagram is commutative:

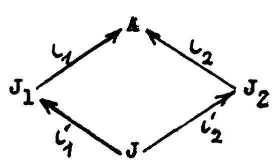


R is obviously reflexive and symmetric. It is also transitive: let $\langle i, \iota_1\rangle R \langle j, \iota_2\rangle R \langle k, \iota_3\rangle$. Then there is a commutative diagram



in which ι'_1, ι'_2, ι''_2 and ι''_3 are s-paths and $\iota'_2(0) = j = \iota''_2(0)$, $\iota'_1(0) = i$, $\iota'_3(0) = k$. Hence $\iota'_2 = \iota''_2$, $J_4 = J_5$ by Lemma 2 and $\iota_1 \iota'_1 = \iota_2 \iota'_2 = \iota_2 \iota''_2 = \iota_3 \iota''_3$. Thus $\langle i, \iota_1\rangle R \langle k, \iota_3\rangle$.

R is a congruence on $P((A,f))$: let $\langle i, \iota_1\rangle R \langle j, \iota_2\rangle$ and $\langle i, \iota_1\rangle \in D(f')$ (then $\langle j, \iota_2\rangle \in D(f')$ because $\iota_1(i) = \iota_2(j)$). Then there is a commutative diagram



in which ι'_1, ι'_2 are s-paths. By Lemma 2 there is an s-path $\iota : J_3 \longrightarrow J$, such that $\iota(0) = 1$. Now define $\iota''_1 = \iota'_1 \circ \iota$, $\iota''_2 = \iota'_2 \circ \iota$. Obviously, $\iota_1 \circ \iota''_1 = \iota_2 \circ \iota''_2$ and $\iota''_1(0) = i + 1$, $\iota''_2(0) = j + 1$. Thus,

$$f'(\langle i, \iota_1\rangle) = \langle i + 1, \iota_1\rangle R \; f'(\langle j, \iota_2\rangle) = \langle j + 1, \iota_2\rangle.$$

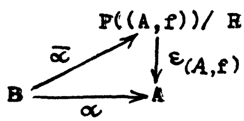
Now, define $\tilde{f}$: $P((A,f))/R \longrightarrow P((A,f))/R$:

1. $[\langle i, \iota \rangle] \in D(\tilde{f})$ iff $\langle i, \iota \rangle \in D(f')$

2. $\tilde{f}([\langle i, \iota \rangle]) = [f'(\langle i, \iota \rangle)] = [\langle i + 1, \iota \rangle]$.

$(P((A,f))/R, \tilde{f})$ is a Pawlak machine, too. Let us define $\varepsilon_{(A,f)}: P((A,f))/R \longrightarrow A$ $\quad \varepsilon_{(A,f)}([\langle i, \iota \rangle]) = \iota(i)$. Obviously, this definition is correct and $\varepsilon_{(A,f)}$ is a simulation of $(P((A,f)/R, \tilde{f})$ in (A,f).

Lemma 5: Let (B,g), (A,f) be Pawlak machines, let α be a simulation of (B,g) in (A,f). Then there is the unique s-homomorphism $\bar{\alpha}: (B,g) \longrightarrow (P((A,f))/R, \tilde{f})$ such that the following diagram is commutative

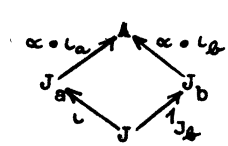


Proof: Let us define $\bar{\alpha}(a) = [\langle 0, \alpha \circ \iota_a \rangle]$ (ι_a is an s-path such that $\iota_a(0) = a$).

1. $\bar{\alpha}$ is an s-homomorphism:

(1) $a \in D(g)$ iff $\alpha(a) \in D(f)$ iff $[\langle 0, \alpha \circ \iota_a \rangle] \in D(\tilde{f})$

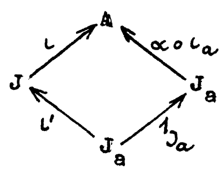
(2) let $a \in D(g)$ and $b = g(a) = \iota_a(1)$. By Lemma 1 we have $j \in J_b$ iff $b \in D(g^j)$ iff $a \in D(g^{j+1})$ iff $j + 1 \in J_a$. Define $\iota : J_b \longrightarrow J_a$, $\iota(i) = i + 1$. Then $\iota_a \circ \iota(0) = \iota_b(0)$. Hence, $\iota_a \circ \iota = \iota_b$ by Lemma 2 and the diagram



is commutative and $\tilde{f}(\bar{\alpha}(a)) = [\langle 1, \iota \circ \iota_a \rangle] = [\langle 0, \iota \circ \iota_b \rangle] = \bar{\alpha}(g(a))$.

Obviously, $\varepsilon_{(A,f)} \circ \overline{\alpha} = \alpha$.

2. Let $\tilde{\alpha}$ be an s-homomorphism such that $\varepsilon_{(A,f)} \circ \tilde{\alpha} = \alpha$. Let $a \in B$, let $\tilde{\alpha}(a) = [\langle i, \iota \rangle]$. By Lemma 1 $j \in J_a$ iff $\tilde{\alpha}(a) \in D(\tilde{f}^j)$, for $\tilde{\alpha} \circ \iota_a$ is an s-path. By the definition of $\tilde{f}$: $\tilde{\alpha}(a) \in D(f^j)$ iff $i + j \in D(\iota)$. Thus, $\iota(i + j) = \varepsilon_{(A,f)} \circ \tilde{\alpha} \circ \iota_a(j) = \alpha \circ \iota_a(j)$. Now, if we define $\iota' : J_a \longrightarrow J$ $\iota'(j) = i + j$, the diagram



is commutative. Hence, $\langle 0, \alpha \circ \iota_a \rangle R \langle i, \iota \rangle$, i.e. $\overline{\alpha}(a) = \tilde{\alpha}(a)$.

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