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Rainald Schöneberg

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ON THE STRUCTURE OF FIXED POINT SETS OF PSEUDO-CONTRACTIVE MAPPINGS II.

Rainald SCHÖNEBERG, Aachen

Abstract: Let $(E, \| \ \|)$ be a (real) normed linear space, X a subset of E and let f map X into E. The present paper investigates the nature of the set of solutions of the equation $f(x) = x$ if f is nonexpansive or (more generally) pseudo-contractive.

Key words: Nonexpansive, pseudo-contractive, metrically convex, pathwise connected, weakly inward.

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1 Preliminaries and notations. In § 1 we introduce definitions of certain concepts to be used in this paper, describe a method which will be helpful to reduce the fixed point problem for pseudo-contractive mappings to the nonexpansive case and establish several properties of this reduction. In § 2 we prove a general result on the structure of the complement of the fixed point set of a pseudo-contractive mapping and state some interesting consequences. § 3 is motivated by the observation that the method of proof used in [15] yields an improved version of the main result of [15]. We state and prove this generalization and use it to deduce a number of new results on the structure of fixed point sets

of nonexpansive and pseudo-contractive mappings in strictly convex and arbitrary Banach-spaces.

All normed linear spaces occurring in this paper are assumed to be real normed linear spaces.

Let $(E, \| \|)$ be a normed linear space. For $X \subset E$ and $f: X \longrightarrow E$ we let $\overline{X}$ denote the closure of X and Fix(f) is defined to be the fixed point set of f. If $H \subset E$ and $X \subset H$ then the symbol $\partial_H X$ stands for the boundary of X in the subspace H. In case $H = E$ we write ∂X instead of $\partial_E X$.

A subset X of E is said to be <u>metrically convex</u> if for each pair of distinct points x_1, x_2 of X there is a point x of X, distinct from x_1 and x_2, such that

$$\| x_1 - x_2 \| = \| x_1 - x \| + \| x_2 - x \| .$$

Every closed and metrically convex subset of a Banach-space is pathwise connected (see [11]). The properties "convex" and "metrically convex" coincide for closed subsets of strictly convex normed linear spaces.

A subset X of E is said to be <u>starshaped</u> if there is $x_0 \in X$ such that $(1 - t)x_0 + tx \in X$ whenever $t \in [0,1]$ and $x \in X$.

It is well-known and easily verified that the path components of the complement of a starshaped set are unbounded.

According to [9] we say that a set $X \subset E$ has <u>normal structure</u> if for every convex and bounded subset K of X which contains more than one point there is a point x in K for which

$$\sup(\{\| x - y \| \mid y \in K\}) < \sup(\{\| u - v \| \mid u, v \in K\}).$$

If X is compact or $(E, \| \|)$ is uniformly convex then X has normal structure.

If $X \subset E$ and $f: X \longrightarrow E$ then f is said to be <u>weakly inward</u> iff

$$\lim_{h \to 0^+} h^{-1} d((1 - h)x + hf(x), X) = 0$$

for each $x \in X$, where $d(z,X) := \inf(\{\| z - y \| \mid y \in X\})$. Note that f is weakly inward if $f[\partial X] \subset X$ and X is convex. f is said to be <u>nonexpansive</u> if for all $x, y \in X$

$$\| f(x) - f(y) \| \leq \| x - y \| ,$$

while f is said to be <u>pseudo-contractive</u> if for all $x, y \in X$ and $r \geq 0$

$$\| x - y \| \leq \| (1 + r)(x - y) - r(f(x) - f(y)) \| .$$

The pseudo-contractive mappings are easily seen to be more general than the nonexpansive mappings. They derive their importance in nonlinear functional analysis via their connection with the accretive transformations: A mapping $f: X \longrightarrow E$ is pseudo-contractive if and only if the mapping $Id - f$ is accretive, i.e., for every $x, y \in X$ there is $j \in J(x - y)$ such that

$$(f(x) - f(y), j) \leq \| x - y \|^2,$$

where $J: E \longrightarrow 2^{E^*}$ denotes the <u>normalized duality mapping</u> which is defined by

$$J(x) := \{ j \in E^* \mid (x, j) = \| x \|^2 \text{ and } \| j \| = \| x \| \}$$

(see [8]).

Our main tool in studying the fixed point problem for pseudo-contractive mappings is

<u>Proposition 1</u>. <u>Let</u> $(E, \| \ \|)$ <u>be a Banach-space</u>, $X \subset E$ <u>and let</u> $f: X \longrightarrow E$ <u>be continuous and pseudo-contractive</u>. <u>Let furthermore</u> $A_f: X \longrightarrow E$ <u>be defined by</u> $A_f := 2Id - f$. <u>Then</u>:

(1) A_f *is one-to-one and* A_f^{-1} *is nonexpansive.*

(2) $\mathrm{Fix}(f) = \mathrm{Fix}(A_f^{-1})$

(3) *If* f *is weakly inward and* X *is closed and convex then* $X \subset A_f[X]$.

(4) *If* X *is closed then* $A_f[X]$ *is closed.*

(5) *If* X *is open then* $A_f[X]$ *is open.*

Proof: (1),(4): For $x,y \in X$ we have by definition

$$\| A_f(x) - A_f(y) \| \geq \| x - y \|$$

which establishes (1) and (4). (2): Obvious. (3): Let z be in X and define $g: X \longrightarrow E$ by $g(x) := \frac{1}{2}(f(x) + z)$. Then g is weakly inward and continuous. Let $x,y \in X$. By the previous remark there is $j \in J(x - y)$ such that $(f(x) - f(y), j) \leq \| x - y \|^2$. This implies $(g(x) - g(y), j) \leq \frac{1}{2} \| x - y \|^2$. Hence by [4, Corollary 2] there is $x \in X$ with $g(x) = x$, i.e., $z = A_f(x)$. (5): Let $x,y \in X$ and choose $j \in J(x - y)$ with $(f(x) - f(y), j) \leq \| x - y \|^2$. Then

$(A_f(x) - A_f(y), j) = 2(x - y, j) - (f(x) - f(y), j) \geq \| x - y \|^2$.

Therefore by [4, Theorem 3] $A_f[X]$ is an open subset of E.

Q.E.D.

2 *Complements of fixed point sets.* The main result of this section is

Theorem 1. Let $(E, \| \ \|)$ *be a normed linear space,* $X \subset E$ *and let* $f: X \longrightarrow E$ *be pseudo-contractive. Then every* $x \in X \setminus \mathrm{Fix}(f)$ *lies in an unbounded path component of* $E \setminus \mathrm{Fix}(f)$.

Proof. Let $x \in X \setminus \mathrm{Fix}(f)$ and define $H \subset E$ by

$$H := \{ x + r(x - f(x)) \mid r \geq 0 \}.$$

Then H is a pathwise connected unbounded subset of E with $x \in H$. Therefore it suffices to show that $H \cap Fix(f) = \emptyset$. Otherwise there is $r > 0$ such that $y := x + r(x - f(x)) \in Fix(f)$. Since f is pseudo-contractive this yields

$$\| x - y \| \leq \| (1 + r)(x - y) - r(f(x) - y) \| = 0.$$

Hence $x = y$, i.e., $x \in Fix(f)$, a contradiction. Q.E.D.

It is well-known that in finite-dimensional normed linear spaces the boundary of a nonempty open and bounded set is not a continuous retract of the closure of this set (see [6]). This isn't true for infinite-dimensional normed linear spaces. Indeed, a normed linear space is infinite-dimensional if and only if there is a continuous retraction of the unit ball onto the unit sphere (see [5]). However, such a retraction cannot be pseudo-contractive. This is a consequence of the following result, which improves a corresponding one due to Floret [7].

Corollary 1.1. If $(E, \| \ \|)$ is a normed linear space and X is a nonempty open and bounded subset of E then there is no pseudo-contractive retraction of $\overline{X}$ onto ∂X.

Proof. If $R: \overline{X} \longrightarrow E$ is pseudo-contractive such that $\partial X \subset Fix(R)$ then $R = Id$ by Theorem 1. Hence R cannot be a retraction onto ∂X. Q.E.D.

Another useful consequence of Theorem 1 is

Corollary 1.2. Let $(E, \| \ \|)$ be a normed linear space and X a subset of E such that the path components of $E \setminus X$ are unbounded. Let furthermore $f: X \longrightarrow E$ be pseudo-contractive. Then every path component of $E \setminus Fix(f)$ is unbounded.

Proof. Let U be a path component of $E \setminus Fix(f)$ and suppose, contrary to our assertion, that U is bounded. Then by Theorem 1 $U \cap X = \emptyset$, i.e., U is a bounded path component of $E \setminus X$, a contradiction. Q.E.D.

Corollary 1.3. Let $(E, \|\ \|)$ *be a normed linear space,* X *a starshaped subset of* E *and let* $f: X \longrightarrow E$ *be pseudo-contractive. Then every path component of* $E \setminus Fix(f)$ *is unbounded.*

Remark 1. In the case of a convex X and a compact and nonexpansive selfmapping of X, Corollary 1.3 was proved by Stoyan in [16].

3 *Fixed point sets.* The results of this section are based on the following improved version of Theorem 1 in [15]. Although the method of proof is the same as in [15] we give the proof for the sake of completeness.

Theorem 2. Let $(E, \|\ \|)$ *be a normed linear space,* $H \subset E$ *be closed and convex,* $X \subset H$ *be closed and let* $f: X \longrightarrow H$ *be nonexpansive. Then* $Fix(f)$ *is closed and metrically convex if the following conditions are satisfied:*

(1) *If* K *is a nonempty, bounded, closed and convex subset of* X *such that* K *lies in some sphere and* $f[K] \subset K$ *then* f *has a fixed point in* K.

(2) $card(Fix(f) \cap \partial_H X) \leq 1$

Proof. Since X is closed and f is continuous the fixed point set of f is closed. Let now $x_1, x_2 \in Fix(f)$ with $x_1 \neq x_2$. Because of (2) we may assume the existence of $r > 0$ such that $y \in H$ and $\|y - x_1\| \leq r$ imply $y \in X$. Choose $t \in (0,1)$ with $t\|x_1 - x_2\| \leq r$ and define $K \subset E$ by

$K := \{y \in H \mid \| x_1 - y \| \leq t \| x_1 - x_2 \|$ and $\| x_2 - y \| \leq$
$\leq (1 - t) \| x_1 - x_2 \| \}$

Then K is a nonempty $[(1 - t)x_1 + tx_2 \in K]$, bounded, closed and convex subset of X such that K lies in the sphere of radius $t \| x_1 - x_2 \|$ about x_1. Using the nonexpansiveness of f it is easily verified that $f[K] \subset K$. Hence by (1) there is $x \in K$ with $f(x) = x$. Now

$$\| x_1 - x \| + \| x_2 - x \| \leq t \| x_1 - x_2 \| + (1 - t) \| x_1 - x_2 \| = \| x_1 - x_2 \|$$

and therefore

$$\| x_1 - x \| + \| x_2 - x \| = \| x_1 - x_2 \| .$$

Since $x \neq x_1$ and $x \neq x_2$ we are done. Q.E.D.

It is well-known that a nonempty convex subset of a strictly convex normed linear space which lies in some sphere consists exactly of one point. Hence (1) of Theorem 2 is always satisfied if $(E, \| \ \|)$ is assumed to be strictly convex. This yields the following extension of the classical result of Schaefer [14]:

Corollary 2.1. Let $(E, \| \ \|)$ be a strictly convex normed linear space, $H \subset E$ be closed and convex, $X \subset H$ be closed and let $f: X \longrightarrow H$ be nonexpansive such that $\mathrm{card}(\mathrm{Fix}(f) \cap \partial_H X) \leq 1$. Then $\mathrm{Fix}(f)$ is closed and convex.

The most famous fixed point theorem for nonlinear nonexpansive mappings in the noncompact setting is that obtained independently by Browder, Göhde and Kirk who proved that a nonexpansive selfmapping of a nonempty weakly compact and convex subset of a Banach-space has a fixed point whenever this set has normal structure (see [9]). This yields, observing Theorem 2

Corollary 2.2. *Let* $(E, \| \, \|)$ *be a Banach-space,* $H \subset E$ *be closed and convex and* $X \subset H$ *be closed such that every closed, bounded and convex subset of* X *is weakly compact and has normal structure. If* $f: X \longrightarrow H$ *is nonexpansive such that* $\text{card}(\text{Fix}(f) \cap \partial_H X) \leq 1$ *then* $\text{Fix}(f)$ *is closed and pathwise connected.*

In the remaining part of this section we treat the "structure-problem" for fixed point sets of pseudo-contractive mappings.

Corollary 2.3. *Let* $(E, \| \, \|)$ *be a strictly convex Banach-space,* $X \subset E$ *be closed and convex and let* $f: X \longrightarrow E$ *be continuous, pseudo-contractive and weakly inward. Then* $\text{Fix}(f)$ *is closed and convex.*

Proof. Define $A_f: X \longrightarrow E$ by $A_f(x) := 2x - f(x)$. Because of Proposition 1, (1) and (3) we may define $g: X \longrightarrow X$ by $g(x) := A_f^{-1}(x)$. Then $\text{Fix}(f) = \text{Fix}(g)$ and g is nonexpansive by Proposition 1, (1) and (2). Now Corollary 2.1 implies that $\text{Fix}(g)$ is closed and convex, which yields the assertion.

Q.E.D.

Corollary 2.4. *Let* $(E, \| \, \|)$ *be a Banach-space and let* X *be a nonempty, weakly compact and convex subset of* E *which has normal structure. Let furthermore* $f: X \longrightarrow E$ *be continuous, pseudo-contractive and weakly inward. Then* $\text{Fix}(f)$ *is nonempty, a nonexpansive retract of* X *) *and, in particu-*

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*) i.e. there is a nonexpansive mapping $r: X \longrightarrow \text{Fix}(f)$ with $r(x) = x$ if $x \in \text{Fix}(f)$. It should be noted that a nonexpansive retract of a convex set is metrically convex (see [3]).

lar, pathwise connected.

Proof. Let A_f and g be defined as in the proof of Corollary 2.3. Then Fix(g) is nonempty by the Browder/Göhde/Kirk theorem and a nonexpansive retract of X by [3, Theorem 2]. Using Fix(f) = Fix(g) we are done. Q.E.D.

Remark 2. The existence part of Corollary 2.4 was proved for Hilbert spaces in [12] and for arbitrary Banach-spaces in [11].

Let $(E, \| \ \|)$ be a normed linear space, $X \subset E$ and $f: X \longrightarrow E$. Recall (see [10]) that f is said to be generalized condensing if whenever $Y \subset X$, $f[Y] \subset Y$ and $Y \setminus \overline{co}(f[Y])$ is relatively compact, then Y is relatively compact, where $\overline{co}(f[Y])$ is the convex closure of $f[Y]$.

Corollary 2.5. Let $(E, \| \ \|)$ be a Banach-space, $X \subset E$ be nonempty, closed, bounded and convex and let $f: X \longrightarrow X$ be continuous, pseudo-contractive and generalized condensing. Then Fix(f) is nonempty, compact and pathwise connected.

Proof. The Lifschitz-Sadovski fixed point theorem [10] implies that Fix(f) is nonempty and compact. Furthermore it is well-known (see [13]) that there is a compact and convex subset Y of X with $Fix(f) \subset Y$ and $f[Y] \subset Y$. Hence Corollary 2.4 applied to $(f|_Y, Y)$ shows that Fix(f) is pathwise connected. Q.E.D.

Corollary 2.6. Let $(E, \| \ \|)$ be a Banach-space such that every nonempty, closed, bounded and convex subset of E which lies in some sphere has the fixed point property with respect to nonexpansive selfmappings. Let furthermore $X \subset E$ be open and $f: \overline{X} \longrightarrow E$ be continuous and pseudo-

contractive such that $card(Fix(f) \cap \partial X) \leq 1$. *Then* $Fix(f)$ *is closed, metrically convex and, in particular, pathwise connected.*

Proof. Define $A_f: \overline{X} \longrightarrow E$ by $A_f(x) := 2x - f(x)$, $Y \subset E$ by $Y := A_f[\overline{X}]$ and $g: Y \longrightarrow E$ by $g(x) := A_f^{-1}(x)$. Proposition 1 implies that Y is closed, g is nonexpansive, $\partial Y \subset A_f[\partial X]$ and that $Fix(g) = Fix(f)$. Hence $card(Fix(g) \cap \partial Y) \leq 1$, which implies - by Theorem 2 - that $Fix(g) = Fix(f)$ is metrically convex. Q.E.D.

Corollary 2.7. Let $(E, \| \ \|)$ *be a strictly convex Banach-space,* $X \subset E$ *open and let* $f: \overline{X} \longrightarrow E$ *be continuous and pseudo-contractive such that* $card(Fix(f) \cap \partial X) \leq 1$. *Then* $Fix(f)$ *is closed and convex.*

Remark 3. The results of this section seem to be new. For the detailed discussion of relevant contributions of other authors and some applications of results similar to those of this section we refer to [15].

The present paper extends several results of an earlier one of the author, which appeared under the same title in Comment. Math. Univ. Carolinae 17,4(1976), 771-777. It should be noted, however, that the proofs in that paper don't use such deep results from the theory of differential equations in Banach spaces as [4, Corollary 2] and [4, Theorem 3].

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Lehrstuhl C für Mathematik

RWTH Aachen

Templergraben 55

D-5100 Aachen

B R D

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