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ONCE MORE ON CONTINUITY OF MAXIMAL MONOTONE MAPPINGS

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Abstract: In the paper there is given an alternate and more elementary proof of the theorem due to Kenderov and Robert, concerning continuity of monotone mappings.

Key words: Banach space, property (H), maximal monotone multivalued mapping, singlevaluedness, upper semicontinuity.

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In the paper by Kenderov and Robert [3], the proof of the following theorem is outlined.

Theorem. Let X be a real Banach space whose dual X^* has the property (H) (see § 0). Let $T: X \longrightarrow 2^{X^*}$ be a maximal monotone multivalued mapping such that int $D(T) \neq \emptyset$.

Then the set of all those $x \in$ int $D(T)$ for which Tx is a singleton and T is (strongly) upper semicontinuous at x (i.e., to every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $u \in D(T)$, fulfilling $\|u - x\| < \delta$, the set Tu is included in the ε-neighbourhood of Tx), is dense residual in int $D(T)$.

The author [1] has received the same conclusion provided that X^* is strictly convex and has the weaker property (H_ω) (see § 0). In this note, adapting the method of [1] and using some ideas of [3], we present an alternate and more elementary proof of Theorem. In doing so we do not need either the local boundedness of T or any results of geometry of Banach spaces, which seem to be used in [3]. Note that they are Lemmas 1.5, 1.6 and 2.2 which have been stimulated by [3].

Our method is based on the simple fact that a maximal monotone multivalued mapping is demiclosed. Therefore, we first study demiclosed mappings, which are far more general than maximal monotone ones. Combining the obtained results with special properties of monotone mappings, we then get Theorem.

§ 0. Preliminary notations. In this note (unless otherwise stated) P will mean a metric space, X a real normed linear space and X^* its topological dual endowed with the norm dual to the norm on X. We shall say that X^* has the property (H) (resp. (H_ω)) if for each net (resp. sequence) $\{w_\alpha\} \subset X^*$ and each $w \in X^*$ the following implication holds

$$(w_\alpha \rightharpoonup w \ \& \ \|w_\alpha\| \longrightarrow \|w\|) \Longrightarrow w_\alpha \longrightarrow w,$$

where the arrow "$\rightharpoonup$" means the weak* convergence in X^*. Obviously, $(H) \Longrightarrow (H_\omega)$. If $A \subset P$, the symbols int A, cl A will stand for the interior, resp. the closure of A.

Let $T: P \longrightarrow 2^{X^*}$ be an arbitrary multivalued mapping from P to X^*. The domain of T will be denoted by D(T). A single-valued mapping $T_1: P \longrightarrow X^*$ having the same domain as T, i.e., $D(T_1) = D(T)$, and such that $T_1 \subset T$ (we do not distinguish between a mapping and its graph) is called a selection of T. Now, define the function $f_T: P \longrightarrow (-\infty, +\infty]$ by

$$f_T(u) = \inf\{\|w\| \mid w \in Tu\}, \ u \in P,$$

the mapping $\overline{T}: P \longrightarrow 2^{X^*}$ by

$$\overline{T} = \{(u,w) \in T \mid \|w\| = f_T(u)\} ,$$

and the following sets (T_1 being a selection of T)

$$SV(T) = \{u \in D(T) \mid \text{ Tu is a singleton}\}$$

$$C(f_T) = \{u \in D(T) \mid f_T \text{ is continuous at } u\}$$

$$C(T_1) = \{u \in D(T) \mid T_1 \text{ is continuous at } u\}$$

$$C^d(T_1) = \{u \in D(T) \mid T_1 \text{ is demicontinuous at } u\} ,$$

where demicontinuity means continuity from the metric topology to the weak* topology.

Finally, let $F: Q \longrightarrow 2^P$ be a multivalued mapping from a topological space Q to a metric space P (with the domain $D(F) = Q$). We recall that F is said to be upper (resp. lower) semicontinuous at $u \in Q$ if to each $\varepsilon > 0$ there exists a neighbourhood V of u such that for every $v \in V$ the set Fv (resp. Fu) is contained in the ε-neighbourhood of the set Fu (resp. Fv). The sets of all the points $u \in D(F)$ at which F is upper (resp. lower) semicontinuous will be denoted by $C_U(F)$ (resp. $C_L(F)$).

§ 1. Throughout the paragraph $T: P \longrightarrow 2^{X^*}$ will denote a demiclosed multivalued mapping, i.e.,

$$\forall u \in P \quad \forall w \in X^* \quad \forall \text{ net } \{(u_\alpha, w_\alpha)\} \subset T$$

$$(u_\alpha \longrightarrow u,\ w_\alpha \rightharpoonup w,\ \sup_\alpha \|w_\alpha\| < +\infty) \Longrightarrow (u,w) \in T.$$

It can be easily seen that

$$D(\overline{T}) = D(T) = \{u \in P \mid f_T(u) < +\infty\} .$$

Lemma 1.1 ([1, Lemma 1.1]): The function f_T is lower semicontinuous, i.e. for any real a the set $\{u \in P \mid f_T(u) \leq \leq a\}$ is closed.

Lemma 1.2 ([1, Lemma 1.2]): The set $C(f_T)$ is residual in $D(T)$.

The following two lemmas are generalizations of Lemmas 1.3 and 1.4 in [1].

Lemma 1.3: If T_0 is an arbitrary selection of $\overline{T}$, then

$$C(f_T) \cap SV(\overline{T}) \subset C^d(T_0).$$

Proof: Let $u \in C(f_T) \cap SV(\overline{T})$ and let $\{u_n\} \subset D(T)$ be a sequence converging to u. Then $f_T(u_n) \longrightarrow f_T(u)$, i.e., $\| T_0 u_n \| \longrightarrow \| T_0 u \|$. Hence the sequence $\{T_0 u_n\}$ is bounded. Let $\{T_0 u_{n_\alpha}\}$ be an arbitrary subnet of $\{T_0 u_n\}$ converging weakly* to some $w \in X^*$. Then, by the demiclosedness of T, $w \in Tu$, and $\| w \| \geq \| T_0 u \|$. On the other hand, the weak* lower semicontinuity (w*.l.s.c. in abbreviation) of the norm on X^* gives $\| w \| \leq \lim_\alpha \inf \| T_0 u_{n_\alpha} \| = \| T_0 u \|$. Thus $\| w \| = \| T_0 u \|$, $w \in \overline{T}u$. And since $u \in SV(\overline{T})$, $w = T_0 u$. So we have shown that $\{T_0 u_{n_\alpha}\}$ converges weakly* to $T_0 u$. But $\{T_0 u_{n_\alpha}\}$ was an arbitrary subnet of $\{T_0 u_n\}$. Therefore $T_0 u_n \rightarrow T_0 u$, too. It means $u \in C^d(T_0)$.

Lemma 1.4: Suppose that X^* has the property (H_ω). Then for any selection T_0 of $\overline{T}$ the following inclusion holds

$$C(f_T) \cap SV(\overline{T}) \subset C(T_0).$$

Proof: It follows immediately from Lemma 1.3.

Lemma 1.5: If X^* has the property (H), then, for each $u \in C(f_T)$, $\overline{T}u$ is a compact set, and $C(f_T) \subset C_U(\overline{T})$.

Proof: Let $u \in C(f_T)$. Let $\{w_\alpha\}$ be a net in $\overline{T}u$. Then $\| w_\alpha \| = f_T(u)$, hence $\{w_\alpha\}$ is weakly* praecompact and, from $\{w_\alpha\}$, we can extract a subnet $\{w_\beta\}$ converging weakly* to some $w \in X^*$. The w*.l.s.c. of the norm on X^* gives $\| w \| \leq \lim_\beta \inf \| w_\beta \| = f_T(u)$. But thanks to the demiclosedness of T, $w \in Tu$, thus $\| w \| \geq f_T(u)$. Therefore $\| w \| = f_T(u)$ and $w \in \overline{T}u$, which proves the compactness of $\overline{T}u$.

Next we shall prove the upper semicontinuity of $\overline{T}$ at $u \in C(f_T)$. Suppose the contrary. Then there is an $\varepsilon > 0$ and a sequence $\{(u_n, w_n)\} \subset \overline{T}$ such that $u_n \longrightarrow u$ but

(*) $\| w_n - \overline{T}u \| = \inf \{ \| w_n - z \| \mid z \in \overline{T}u \} \geq \varepsilon > 0,\ n = 1,2,\dots$.

Since $u \in C(f_T)$, $\| w_n \| = f_T(u_n)$ converge to $f_T(u)$, so the sequence $\{w_n\}$ is bounded, i.e., weakly* praecompact. Therefore, there is a subnet $\{w_{n_\alpha}\} \subset \{w_n\}$ and $w \in X^*$ so that $w_{n_\alpha} \rightharpoonup w$. The demiclosedness of T gives $w \in Tu$ and since $\| w \| \leq \lim_\alpha \inf \| w_{n_\alpha} \| = f_T(u)$, w belongs to $\overline{T}u$. Thus we have $w_{n_\alpha} \rightharpoonup w$ and $\| w_{n_\alpha} \| \longrightarrow \| w \|$. Now, the property (H) yields $w_{n_\alpha} \longrightarrow w \in \overline{T}u$, which contradicts (*). So the upper semicontinuity of $\overline{T}$ at u is proved, i.e., $u \in C_U(\overline{T})$, and hence $C(f_T) \subset C_U(T)$.

Proposition 1.1 ([2]): Let $F: Q \longrightarrow 2^P$ be a multivalued mapping from a topological space Q to a metric space P (with $D(F) = Q$) such that $C_U(F) = Q$ and that for each $u \in Q$ the set Fu is compact. Then the set $C_L(F)$ is residual in Q.

Taking $Q = C(f_T)$, $P = X^*$ and $F = \overline{T}/_{C(f_T)}$ (restriction of $\overline{T}$ on the set $C(f_T)$), we see, by Lemma 1.5, that the hypotheses of Proposition 1.1 are fulfilled. Hence

<u>Lemma 1.6</u>: If X^* has the property (H), then the set $C_L\left(\overline{T}/_{C(f_T)}\right)$ is residual in $C(f_T)$.

§ 2. Recall that a mapping $T: X \longrightarrow 2^{X^*}$ is said to be monotone if

$$\forall (x,x^*) \in T \ \forall (y,y^*) \in T \ \langle x^* - y^*, x - y \rangle \geq 0,$$

where $\langle .,. \rangle$ denotes the duality pairing between X^* and X, and maximal monotone if there is no proper monotone extension of T. In what follows we shall assume that $T: X \longrightarrow 2^{X^*}$ is a maximal monotone multivalued mapping such that $\operatorname{int} D(T) \neq \emptyset$.

<u>Lemma 2.1</u> (see [1, Lemma 2.1]): T is demiclosed.

<u>Lemma 2.2</u>: Let $T': X \longrightarrow 2^{X^*}$ be a monotone multivalued mapping such that $\operatorname{int} (\operatorname{cl} D(T')) \neq \emptyset$. Then

$$C_L(T') \cap \operatorname{int} (\operatorname{cl} D(T')) \subset SV(T').$$

<u>Proof</u>: Let $x \in C_L(T') \cap \operatorname{int} (\operatorname{cl} D(T'))$. Suppose there are two different elements w_1, w_2 in $T'x$. Choose $y \in X$ so that

$$3/4 < \| y \| < 1, \ \langle w_1 - w_2, y \rangle \geq \tfrac{1}{2} \| w_1 - w_2 \| > 0$$

and take a positive $\varepsilon < \frac{1}{8} \|w_1 - w_2\|$. Since T' is lower semicontinuous at x, there is a $\delta > 0$ such that

$$(u \in D(T'), \ \|x - u\| < \delta) \Longrightarrow T'x \subset U_\varepsilon (T'u),$$

where $U_\varepsilon(M)$ means the ε-neighbourhood of M. δ can be assumed so small that $\| x - u \| < \delta$ implies $u \in \mathrm{cl}\, D(T')$. Then $x + \delta y \in \mathrm{cl}\, D(T')$ and there exists $u_0 \in D(T')$ such that $\| x + \delta y - u_0 \| < \delta(1 - \| y \|)$. Hence

$$\| x - u_0 \| \leq \| x + \delta y - u_0 \| + \| \delta y \| < \delta ,$$

from which we get $T'x \subset U_\varepsilon(T'u_0)$. Therefore, we can find $z_0 \in T'u_0$ such that

$$\| w_2 - z_0 \| < 2\varepsilon < \frac{1}{4} \| w_1 - w_2 \| .$$

Now, from the monotonicity of T', we have

$$0 \leq \langle z_0 - w_1, u_0 - x \rangle = \langle z_0 - w_2, u_0 - x \rangle +$$

$$+ \langle w_2 - w_1, u_0 - (x + \delta y) \rangle + \langle w_2 - w_1, \delta y \rangle \leq$$

$$\leq \| w_2 - z_0 \| \, \| u_0 - x \| + \| w_2 - w_1 \| \cdot \| u_0 - (x + \delta y) \| +$$

$$+ \delta \langle w_2 - w_1, y \rangle <$$

and using the previous inequalities

$$< \| w_1 - w_2 \| \, (\delta/4 + \delta(1 - \| y \|) - \delta/2) < 0,$$

which is impossible. $T'x$ is thus a singleton, i.e., $x \in SV(T')$.

<u>Proposition 2.1</u>: Let X be a real Banach space whose dual X^* has the property (H). Then the set $SV(\overline{T}) \cap \mathrm{int}\, D(T)$ is dense residual in int D(T).

<u>Proof</u>: Denote $T' = \overline{T}/_{C(f_T)}$. Thanks to Lemma 1.2, the set $C(f_T)$ is residual in D(T). Hence, by Baire's category theorem, $\mathrm{int}\, D(T) \subset \mathrm{cl}\, C(f_T)$. Thus

$$\mathrm{int}\, (\mathrm{cl}\, D(T')) = \mathrm{int}\, (\mathrm{cl}\, C(f_T)) \supset \mathrm{int}\, D(T)$$

and, according to Lemma 2.2,

$$C_L(T')\cap \text{int } D(T)\subset SV(T')\cap \text{int } D(T).$$

But $C_L(T')$ is residual in D(T) since $C_L(T')$ is residual in $C(f_T)$ (Lemma 1.6) and $C(f_T)$ is residual in D(T) (lemma 1.2). Therefore the last inclusion implies that the set $SV(T')\cap$ $\cap$ int D(T) is residual in int D(T). Now, Baire's theorem and the obvious inclusion $SV(T')\subset SV(\overline{T})$ complete the proof.

<u>Lemma 2.3</u> ([1,Lemma 2.2]): Let $T': X\longrightarrow 2^{X^*}$ be a monotone multivalued mapping with int $D(T')\neq\emptyset$ and let T_1' be an arbitrary selection of T'. Then

$$C^d(T_1')\cap \text{int } D(T')\subset SV(T').$$

<u>Proposition 2.2</u>: Let X be a real Banach space whose dual X^* has the property (H). Then the set $SV(T)\cap$ int D(T) is dense residual in int D(T).

<u>Proof:</u> It follows from Proposition 2.1 and Lemma 1.2 that the set $SV(\overline{T})\cap C(f_T)\cap$ int D(T) is residual in int D(T), and, by Lemma 1.3, so is $C^d(T_o)\cap$ int D(T), where T_o denotes a selection of $\overline{T}$. Now, Lemma 2.3 and Baire's theorem yield the conclusion of the proposition.

It should be noted that Proposition 2.2 follows immediately from Proposition 2.1 if we use the fact (see [3]) that, for each $x\in C(f_T)$, $Tx = \overline{T}x$.

<u>Proposition 2.3</u>: Let X be a real Banach space whose dual X^* has the property (H) and let T_o be an arbitrary selection of $\overline{T}$. Then the set $C(T_o)\cap$ int D(T) is dense residual in int D(T).

Proof: Combining Lemmas 1.4, 1.2, Proposition 2.1 and Baire's theorem.

Lemma 2.4 ([1, Lemma 2.3]): If T_1', T_2' are two arbitrary selections of a monotone multivalued mapping $T': X \longrightarrow 2^{X^*}$, with int $D(T') \neq \emptyset$, then

$$C(T_1') \cap \text{int } D(T') = C(T_2') \cap \text{int } D(T').$$

Theorem 2.1 (Kenderov, Robert [3]): Let X be a real Banach space whose dual X^* has the property (H) (where nets are taken). Let $T: X \longrightarrow 2^{X^*}$ be a maximal monotone multivalued mapping such that int $D(T) \neq \emptyset$. Then the set of all those $x \in$ int $D(T)$ for which Tx is a singleton and T is upper semicontinuous at x (i.e., the set $C_U(T) \cap SV(T) \cap \text{int } D(T)$), is dense residual in int D(T).

Proof: It follows from Proposition 2.3 and Lemmas 2.3 and 2.4 in the same way as in the proof of [1, Theorem 2.3].

It should be noted that the set from the above theorem is G_δ, and that the remarks similar to those in [1] hold.

References

[1] M. FABIAN: On singlevaluedness and (strong) upper semicontinuity of maximal monotone mappings, Comment. Math.Univ. Carolinae 18(1977), 19-39.

[2] M.K. FORT: Points of continuity of semicontinuous functions, Public. Math. Debrecen 2(1951-1952),100-102.

[3] P. KENDEROV, R. ROBERT: Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach, Comptes Rendus Acad.Sci. Paris 282(1976), 845-847.

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