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FINITE KILLING VECTOR FIELDS

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Abstract: We produce a new integral formula for tangent vector fields on a Riemannian manifold. By eans of it, we prove a vanishing theorem for finite Killing vector fields, which are the finite analog of classical Killing vector fields.

Key words: Riemannian manifold, Killing vector field, Ricci curvature, integral formula.

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Let (M, ds^2) be an orientable Riemannian manifold, $\dim M = n$; let ∂M denote the boundary of M. On a suitable domain $U \subset M$, choose the 1-forms ω^i $(i,j,\ldots = 1,\ldots,n)$ such that

(1) $$ds^2 = \delta_{ij}\omega^i \omega^j.$$

Then there are 1-forms ω_i^j determined by the conditions

(2) $$d\omega^i = \omega^j \wedge \omega_j^i, \quad \omega_i^j + \omega_j^i = 0;$$

the curvature tensor is given by

(3) $$d\omega_i^j = \omega_i^k \wedge \omega_k^j - \frac{1}{2} R^j_{ik\ell}\, \omega^k \wedge \omega^\ell, \quad R^j_{ik\ell} + R^j_{i\ell k} = 0,$$

the Ricci tensor by

(4) $$R_{ij} = R^k_{ijk}.$$

Further, let $(v_1,\dots,v_n)$ be orthonormal frames in U dual to $(\omega^1,\dots,\omega^n)$. The Euclidean connection of M is then given by the equations

$$\nabla m = \omega^i v_i, \quad \nabla v_i = \omega_i^j v_j. \tag{5}$$

On M, be given a tangent vector field v; in U, let us write

$$v = x^i v_i . \tag{6}$$

The covariant derivatives of x^i with respect to the coframes (ω^i) be defined by

$$dx^i + x^j \omega_j^i = x^i{}_{;j}\, \omega^j. \tag{7}$$

Then

$$(dx^i_{;j} - x^i_{;k}\omega_j^k + x^j_{;j}\omega_k^i) \wedge \omega^j = -\frac{1}{2} x^j R^i_{jk\ell}\, \omega^k \wedge \omega^\ell, \tag{8}$$

and we get the existence of the second order covariant order covariant derivatives $x^i{}_{jk}$ such that

$$dx^i_{;j} - x^i_{;k}\, \omega_j^k + x^k_{;j}\, \omega_k^i = x^i_{;jk}\, \omega^k, \tag{9}$$

$$x^i_{;jk} - x^i_{;kj} = -x^\ell R^i_{\ell kj}. \tag{10}$$

On U, consider the 1-forms

$$\varphi_1 = \partial_{ij} x^k x^j_{;k}\, \omega^i, \quad \varphi_2 = \partial_{ij} x^j x^k_{;k}\, \omega^i; \tag{11}$$

it is easy to see that both φ_1 and φ_2 are globally defined over all of M. The usual $*$-operator be defined by

$$*\omega^i = (-1)^{i+1}\, \omega^1 \wedge \dots \wedge \omega^{i-1} \wedge \omega^{i+1} \wedge \dots \wedge \omega^n, \tag{12}$$

i.e. $\omega^i \wedge *\omega^i = \omega^1 \wedge \ldots \wedge \omega^n =: dv.$

Then

(13) $$d*\varphi_1 = (x^i_{;j}x^j_{;i} + x^i x^j_{;ij})dv,$$

$$d*\varphi_2 = (x^i_{;i}x^j_{;j} + x^i x^j_{;ji})dv$$

and, using (10),

(14) $$d*(\varphi_1 - \varphi_2) = (X - R_{ij}x^i x^j)dv,$$

$$X := x^i_{;i}x^j_{;j} - x^i_{;j}x^j_{;i} = 2\sum_{i<j}(x^i_{;i}x^j_{;j} - x^j_{;i}x^i_{;j}).$$

Our starting point is then the obvious <u>integral formula</u>:

(15) $$\int_{\partial M} *(\varphi_1 - \varphi_2) = \int_M (X - R_{ij}x^i x^j)dv.$$

To the vector field v, associate the quadratic differential form

(16) $$Q_v = \langle \nabla(m + v), \nabla(m + v) \rangle =$$

$$= \partial_{ij}(\partial^i_k + x^i_{;k})(\partial^j_\ell + x^j_{;\ell})\omega^k\omega^\ell$$

on M. The vector field v is said to be a <u>finite Killing vector field</u> if $Q_v = ds^2$, i.e.,

(17) $$\partial_{ij}(\partial^i_k + x^i_{;k})(\partial^j + x^j_{;\ell}) = \partial_{k\ell}$$

or, equivalently, $\| \partial^j_i + x^j_{;i} \|$ is an orthogonal matrix.

<u>Theorem</u>. Let (M,ds^2) be an orientable Riemannian manifold and v a finite Killing vector field on M. Suppose: (i) the Ricci curvature form $R_{ij}\xi^i\xi^j$ of (M,ds^2) is

negative definite on M; (ii) $v = 0$ on ∂M. Then $v = 0$ on M.

Proof. All we have to prove is to establish the inequality $X \geqq 0$ on M and to apply the integral formula (15). Let $m \in M$ be a fixed point. The matrix $\| \sigma_i^j + x^j_{;i} \|$ being orthogonal, there are orthonormal frames at m such that our matrix takes the canonical Jordan form, i.e.,

$$(18) \qquad x^{2i-1}_{;2i-1} = \cos \alpha_i - 1, \; x^{2i}_{;2i-1} = -\sin \alpha_i,$$

$$x^{2i-1}_{;2i} = \sin \alpha_i, \; x^{2i}_{;2i} = \cos \alpha_i - 1 \text{ for } i = 1,\dots,P;$$

$$x^i_{;i} = -2 \text{ for } i = 2P + 1,\dots,2P + R;$$

$$x^j_{;i} = 0 \text{ otherwise.}$$

Then

$$(19) \qquad X = 2(1 + 2R) \sum_{1 \leq \rho \leq P} (1 - \cos \alpha_\rho) +$$

$$+ 4 \sum_{1 \leq \rho < \sigma \leq P} (1 - \cos \alpha_\rho)(1 - \cos \alpha_\sigma) + 2R(R - 1) \geqq 0,$$

and we are done.

Remark. To v, associate the quadratic differential form

$$(20) \quad Q_s' = \langle \nabla m, \nabla (m + v) \rangle = \{ \sigma_{ij} + \frac{1}{2} (\sigma_{ik} x^k_{;j} +$$

$$+ \sigma_{kj} x^k_{;i}) \} \omega^i \omega^j$$

on M. The condition $Q_v' = ds^2$ characterizes the Killing vector fields because it is equivalent to

$$(21) \qquad \sigma_{ik} x^k_{;j} + \sigma_{kj} x^k_{;i} = 0.$$

Using the same integral formula (15), we are in the position to prove the validity of our Theorem even for Killing vector fields(this result being, of course, classical). Indeed, we have just to prove that X is non-negative. Multiplying (21) by σ^{ij}, we get $x^{i}{}_{;i} = 0$. Further, (21) implies $x^{i}{}_{;j} + \sigma^{ik} \sigma_{\ell j} x^{\ell}{}_{;k} = 0$, and we get

$$(22) \qquad X = \sigma^{ik} \sigma_{\ell j} x^{\ell}{}_{;k} \, x^{j}{}_{;i} = x_{j;k} x^{j;k} \geqq 0$$

using the usual notation.

Reference

[1] BOCHNER S., YANO K.: Curvature and Betti numbers, Princeton Univ. Press, 1973.

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