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# ON SINGLEVALUEDNESS AND (STRONG) UPPER SEMICONTINUITY OF MAXIMAL MONOTONE MAPPINGS

Marián FABIAN, Praha

Abstract: Under suitable assumptions on the geometry of a dual $X^*$ of a real Banach space X it is shown that a maximal monotone multivalued mapping T from X to $X^*$ with int $D(T) \neq \emptyset$ is singlevalued and upper semicontinuous on a dense residual subset of int D(T).

Key words: Banach space, demiclosed multivalued mapping, singlevaluedness, upper semicontinuity, differentiability.

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Introduction. Let X be a real Banach space with a topological dual $X^*$, $T: X \longrightarrow 2^{X^*}$ a maximal monotone multivalued mapping whose domain has nonempty interior, i.e., int $D(T) \neq \emptyset$. Two theorems are the main result of this paper, which we can formulate roughly as follows:

Theorem A (on singlevaluedness of T). If the dual $X^*$ is strictly convex, then the set

$MV(T) = \{x \in D(T) \mid T(x) \text{ is not a singleton}\}$

if of the first (Baire's) category in X.

Theorem B (on (strong) upper semicontinuity of T). If the dual $X^*$ is strictly convex and has the property (H) (i.e., if $\{w_n\} \subset X^*$ converges weakly* to $w \in X^*$ and $\|w_n\| \longrightarrow \|w\|$, then $w_n \longrightarrow w$), then there exists a set $C \subset \operatorname{int} D(T)$ dense residual in int D(T) such that for every $x \in C$ the set T(x) is a singleton and T is upper semicontinuous at x, i.e., for $u \in D(T)$ sufficiently close to x, the set T(u) lies in an arbitrary small given (norm) neighbourhood of T(x).

See for details Theorems 2.1 - 2.3, Remarks 2.2 - 2.4 and the definition formulas (2.1) and (2.2).

Let us recall that the property of a mapping $T: X \longrightarrow 2^{X^*}$ to be maximal monotone is independent of which equivalent norm is taken in X. Hence, by using the renorming statement of Amir and Lindenstrauss [2], we obtain that the conclusion of Theorem A holds for any WCG X, especially, for X reflexive or separable. It follows from the renorming statement of John and Zizler [7] that the conclusion of Theorem B is valid for such WCG X which have a WCG dual $X^*$ (more generally, for those WCG X which have an equivalent Fréchet differentiable norm, see [8]), especially, for X reflexive or such X whose dual $X^*$ is separable.

Using the simple fact that a subdifferential of a convex lower semicontinuous function is a monotone multivalued mapping, we get, from Theorems A and B, the well-known results of Asplund [3] concerning the Gâteaux and Fréchet differentiability of convex functions, see Remark 2.6.

The theorem on singlevaluedness of T for X separable has been proved by Zarantonello [21] in a geometrical way, later, topologically, by Kenderov [12] and Robert [16] and more generally, for $X$ with a strictly convex dual $X^*$ by Kenderov [10]. Our Theorem 2.1 is a little improvement of Kenderov's result [10], where it is supposed $D(T) = X$.

The theorem dealing with (strong) upper semicontinuity of $T$ for $X$ with a separable dual $X^*$ has been proved by Robert [17].

The present paper was stimulated by the ideas of Kenderov [10], by means of which he derives the theorem on singlevaluedness of $T$. In doing so he uses the well-known deep fact that $T$ is weakly* upper semicontinuous at each $x \in \in \operatorname{int} D(T)$. However, one can do with the demiclosedness of $T$ only, which is a simple property of maximal monotone mappings.

In this paper, the ideas of Kenderov [10] are generalized to demiclosed multivalued mappings from a metric space $P$ to a dual $X^*$ (see Lemmas 1.1 - 1.3) and extended to the study of the (strong) continuity of such mappings (see Lemma 1.4), and so we get the topological means to prove Theorems 2.1 - 2.3.

The method proposed can be also used for the study of maximal accretive mappings (see, e.g.,[13] for definition).

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§ 0. Preliminaries. Let $U$, $V$ be arbitrary sets. Then each nonempty subset $T$ of $U \times V$ is called a multivalued mapping from $U$ to $V$ and we write $T: U \longrightarrow 2^V$. The set $T^{-1} = \{(v,u) \in V \times U \mid (u,v) \in T\}$ is called the inverse multivalued mapping to $T$. Thus $T^{-1}: V \longrightarrow 2^U$. Obviously, $(T^{-1})^{-1} = T$.

For each $u \in U$, we set

$$T(u) = \{ v \in V \mid (u,v) \in T \} .$$

If the set $T(u)$ consists of one point only, we denote this point by the symbol $T(u)$, too. The set

$$D(T) = \{ u \in U \mid T(u) \neq \emptyset \}$$

is called the domain of $T$, the set $R(T) = D(T^{-1})$, the range of $T$. It is introduced by many authors the graph $G(T)$ of a multivalued mapping $T$ by

$$G(T) = \{ (u,v) \in U \times V \mid v \in T(u) \} .$$

Obviously, $G(T)$ coincides with $T$. Therefore we shall not distinguish between a multivalued mapping and its graph.

A subset $T \subset U \times V$ is called a singlevalued mapping, if the following implication holds:

$$(u,v_1),\ (u,v_2) \in T \Longrightarrow v_1 = v_2.$$

In this case, we write $T\colon U \longrightarrow V$.

A subset $T_1 \subset T \subset U \times V$ is called a selection of the multivalued mapping $T$, if $T_1$ is singlevalued and $D(T_1) = D(T)$.

Throughout the paper $R$ will denote the set of real numbers endowed with the usual topology, $X$ a real normed linear space, $X^*$ its topological dual (the norm on $X^*$ is dual to the norm on $X$), $P$ a metric space. If $A$ is a subset of $P$, then int $A$ will denote the topological interior of $A$ and cl $A$ the closure of $A$. We recall that a subset $A \subset P$ is called residual in $P$ if the set $P \setminus A$ is of the first (Baire's) category in $P$. The arrows " $\longrightarrow$ ", " $\rightharpoonup$ " will denote the strong and weak$^*$ convergence, respectively.

A singlevalued mapping $f\colon P \longrightarrow R \cup \{+\infty\}$ is called a function. The set

$$\operatorname{dom} f = \{ u \in D(f) \mid f(u) < +\infty \}$$

is called the effective domain of f.

A function f is said to be lower semicontinuous if

$$\forall a \in R \,[\text{the set } \{ u \in P \mid f(u) \leq a \} \text{ is closed}]$$

Let $T: P \longrightarrow X^*$ be a singlevalued mapping from a metric space P to a dual $X^*$ and let $u \in D(T)$. T is said to be demicontinuous at u if

$$\forall \text{ sequence } \{u_n\} \subset D(T) \; [u_n \longrightarrow u \Longrightarrow T(u_n) \rightharpoonup T(u)] \,,$$

Let $T: P \longrightarrow 2^{X^*}$ be a multivalued mapping from a metric space P to a dual $X^*$. T is said to be demiclosed if

$$\forall u \in P \; \forall w \in X^* \; \forall \text{ net } \{(u_\alpha, w_\alpha), \alpha \in \Lambda\} \subset T$$

$$[(u_\alpha \longrightarrow u(\Lambda),\; w_\alpha \rightharpoonup w(\Lambda),\; \sup\{\|w_\alpha\| \mid \alpha \in \Lambda\} < +\infty) \Longrightarrow$$

$$\Longrightarrow (u,w) \in T] \,.$$

Let $T: X \longrightarrow 2^{X^*}$ be a multivalued mapping from a real normed linear space X to its dual $X^*$. T is said to be monotone if (for $x \in X$ and $x^* \in X^*$ the symbol $\langle x^*, x \rangle$ denotes the value of the functional $x^*$ at x)

$$\forall (x,x^*) \in T \;\; \forall (y,y^*) \in T \; [\langle x^* - y^*, x - y \rangle \geq 0] \,,$$

and maximal monotone if T is not properly contained in any other monotone mapping.

## § 1. <u>Lemmas on continuity of demiclosed mappings</u>.

<u>Lemma 1.1</u>. Let $T: P \longrightarrow 2^{X^*}$ be a demiclosed multivalued mapping from a metric space P to a dual $X^*$ of a normed linear space X.

Then the function $f_T: P \longrightarrow R \cup \{+\infty\}$ defined by

$$f_T(u) = \inf \{ \| w \| \mid w \in T(u) \}, \quad u \in P \tag{1.1}$$

is lower semicontinuous.

<u>Proof</u>: Let $a \in R$ be arbitrary. We have to show that the set

$$A = \{ u \in P \mid f_T(u) \leqq a \}$$

is closed. Let $u \in \mathrm{cl}\, A$ and let $\{u_n\} \subset A$ be a sequence such that $u_n \longrightarrow u$. For each $n = 1,2,\ldots$, we find $w_n \in T(u_n)$ such that

$$f_T(u_n) \leqq \| w_n \| < f_T(u_n) + 1/n.$$

Thus

$$\| w_n \| < a + 1/n, \quad n = 1,2,\ldots \tag{1.2}$$

and so the sequence $\{w_n\}$ is bounded, hence $w^*$-praecompact. Therefore there is $w \in X^*$ and a subnet $\{w_{n_\alpha}, \alpha \in \Lambda\}$ of the sequence $\{w_n\}$ such that

$$w_{n_\alpha} \longrightarrow w(\Lambda). \tag{1.3}$$

And since $u_{n_\alpha} \longrightarrow u(\Lambda)$, too, and $T$ is demiclosed, $(u,w) \in T$. From the weak* lower semicontinuity ($w^*$.l.s.c., in abbreviation) of the norm on $X^*$, by using (1.2) and (1.3), we have

$$\| w \| \leqq \liminf_{\alpha \in \Lambda} \| w_{n_\alpha} \| \leqq a.$$

Thus $f_T(u) \leqq \| w \| \leqq a$, i.e., $u \in A$. The closedness of $A$ is proved, which completes the proof. Q.E.D.

We recall two well-known propositions.

Proposition 1.1 ([5, 13.4]). If $S: P \longrightarrow Q$ is a single-valued mapping from a metric space P to a metric space Q, then the set C(S) of all those points at which S is continuous, is $G_\delta$ in D(S), i.e., the set $NC(S) = D(S) \setminus C(S)$ is $F_\sigma$ in D(S).

Proposition 1.2. Let P be a metric space and $f: P \longrightarrow R \cup \{+\infty\}$ a lower semicontinuous function. Then the set C(f) of all those points at which f is continuous, is residual in dom f, i.e., the set $NC(f) = \mathrm{dom}\, f \setminus C(f)$ is of the first (Baire's) category in dom f.

Proof: See 14.7.6 and 14.5.2 in [5].

Lemma 1.2. Let $T: P \longrightarrow 2^{X^*}$ be a demiclosed multivalued mapping from a metric space P to a dual $X^*$ of a normed linear space X. Let the function $f_T$ be defined by (1.1).

Then the set $C(f_T)$ of all those points at which $f_T$ is continuous, is residual $G_\delta$ in D(T).

Proof: It follows immediately from Lemma 1.1 and Propositions 1.1 and 1.2.

Let $T: P \longrightarrow 2^{X^*}$ be a multivalued mapping. A selection $T_0$ of T is said to be lower (with respect to the norm on $X^*$), if

$$(1.4) \qquad (u,w) \in T \Longrightarrow \| T_0(u) \| \leq \| w \| .$$

Obviously,

$$(1.5) \qquad \| T_0(u) \| = f_T(u) \text{ for } u \in D(T).$$

We shall show that if T is demiclosed, then there exists at least one lower selection of T. Let $u \in D(T)$ be arbitrary. Denote $c = \inf\{\|w\| \mid w \in T(u)\}$ and set

$$K = \{w \in T(u) \mid \|w\| \leq c + 1\}.$$

Then K is a nonempty bounded and $w^*$-closed subset of $X^*$, hence $w^*$-compact. So the norm on $X^*$, which is $w^*$.l.s.c., attains its minimum on K, i.e., there is a $w_0 \in K \subset T(u)$ such that $\|w_0\| = c$.

For every singlevalued mapping $S: P \longrightarrow X^*$, we introduce the sets

$C^d(S) = \{u \in D(S) \mid S$ is demicontinuous at $u\}$,

$NC^d(S) = D(S) \setminus C^d(S)$.

Lemma 1.3. Let $T: P \longrightarrow 2^{X^*}$ be a demiclosed multivalued mapping from a metric space P to a dual $X^*$ of a normed linear space X, $f_T$ the function defined by (1.1). Let there exist a unique lower selection $T_0$ of T.

Then, if $f_T$ is continuous at $u \in D(T)$, $T_0$ is demicontinuous at u:

(1.6) $$C(f_T) \subset C^d(T_0), \text{ i.e., } NC^d(T_0) \subset NC(f_T)$$

and hence, the set $C^d(T_0)$ is residual in D(T).

Proof: Let $u \in C(f_T)$ be arbitrary. Let $\{u_n\}$ be a sequence in D(T) such that $u_n \longrightarrow u$. Since (1.5) holds and $u \in C(f_T)$,

(1.7) $$\|T_0(u_n)\| \longrightarrow \|T_0(u)\|,$$

hence, the sequence $\{T_0(u_n)\}$ is bounded. It implies that from any subsequence of $\{T_0(u_n)\}$, we can extract a subnet converging weakly* to some $w \in X^*$. Then the demiclosedness of T gives that $(u,w) \in T$, hence, by (1.4), $\| w \| \geq \| T_0(u) \|$. But $w^*$.l.s.c. of the norm on $X^*$, and (1.7) implies $\| w \| \leqq \| T_0(u) \|$. Thus $\| w \| = \| T_0(u) \|$. From here, and from the uniqueness of the lower selection of T, we obtain $w = T_0(u)$. It means that the whole sequence $\{T_0(u_n)\}$ is converging weakly* to $T_0(u)$, so that $u \in C^d(T_0)$, which proves (1.6). Finally, it follows from (1.6), by Lemma 1.2, that the set $C^d(T_0)$ is residual in D(T). Q.E.D.

Corollary 1.1. Let $S: P \longrightarrow X^*$ be a demiclosed single-valued mapping from a metric space P to a dual $X^*$ of a normed linear space X.

Then the set $C^d(S)$ of all those points at which S is demicontinuous, is residual in D(S).

Lemma 1.4. Let P be a metric space and X a normed linear space whose dual $X^*$ has the property (H). Let $T: P \longrightarrow 2^{X^*}$ be a demiclosed multivalued mapping and let there exist a unique lower selection $T_0$ of T. Let $f_T$ be the function defined by (1.1).

Then $T_0$ is continuous at $u \in D(T)$ iff $f_T$ is continuous at u:

(1.8) $$C(T_0) = C(f_T),\ \text{i.e.,}\ NC(T_0) = NC(f_T)$$

and hence, the set $C(T_0)$ is residual $G_\delta$ in D(T).

Proof: Since $X^*$ has the property (H), for every $w \in X^*$ and for every sequence $\{w_n\} \subset X^*$, the following equivalence holds

(1.9) $$w_n \longrightarrow w \Longleftrightarrow (w_n \rightharpoonup w \text{ and } \| w_n \| \longrightarrow \| w \|).$$

Let $u \in D(T)$ and let $\{u_n\}$ be a sequence in $D(T)$ such that $u_n \longrightarrow u$. If we set $w = T_o(u)$ and $w_n = T_o(u_n)$, $n = 1,2,\ldots$ in (1.9), we obtain

$$T_o(u_n) \longrightarrow T_o(u) \Longleftrightarrow (T_o(u_n) \rightharpoonup T_o(u) \text{ and } \| T_o(u_n) \| \longrightarrow \\ \longrightarrow \| T_o(u) \| ).$$

Therefore (see (1.5)),

$$C(T_o) = C(f_T) \cap C^d(T_o).$$

But, by Lemma 1.3, we have $C(f_T) \subset C^d(T_o)$, thus (1.8) holds. The rest of the conclusion of the Lemma follows from the identity (1.8) by Lemma 1.2. Q.E.D.

Corollary 1.2. Let P be a metric space, X a normed linear space whose dual $X^*$ has the property (H). Let $S: P \longrightarrow X^*$ be a demiclosed singlevalued mapping.

Then the set C(S) of all those points at which S is continuous, is residual $G_\delta$ in D(S).

Corollary 1.3. Let $S: P \longrightarrow X$ be a demiclosed singlevalued mapping from a metric space P to a reflexive Banach space X. Then the set C(S) of all those points at which S is continuous, is residual $G_\delta$ in D(S).

Proof: It follows immediately from the renorming statement of Troyanski [20] by Corollary 1.2, where we write $X^*$ instead of X.

It should be noted that, in the book of Alexiewicz [1, V.2.1.], there is a similar statement for X separable:

Let $S: P \longrightarrow X$ be a singlevalued mapping (with D(S) = = P) from a complete metric space P to a separable normed linear space X such that

$$u_n \longrightarrow w \Longrightarrow \langle x^*, S(u_n)\rangle \longrightarrow \langle x^*, S(u)\rangle \text{ for every } x^* \in Z^*,$$

where $Z^*$ is such a subset of $X^*$ that for every $x \in X$,

$$\|x\| = \sup\{\langle x^*, x\rangle \mid x^* \in Z^*, \|x^*\| \leq 1\}.$$

Then the set NC(S) of all those points at which S is not continuous, is of the first category in P.

If $S: Y \longrightarrow X$ is a singlevalued linear closed (i.e., $y_n \longrightarrow y$ and $S(y_n) \longrightarrow x$ imply $y \in D(S)$ and $x = S(y)$) mapping from a normed linear space Y to a reflexive Banach space X, with D(S) of the second category in itself, we receive from Corollary 1.3 with help of Mazur's theorem that S is continuous, which is a special case of Banach's closed graph theorem.

## § 2. Theorems on singlevaluedness and (strong) upper semicontinuity of maximal monotone mappings

We start by the following simple lemma:

Lemma 2.1. A maximal monotone multivalued mapping $T: X \longrightarrow 2^{X^*}$ from a normed linear space X to its dual $X^*$ is demiclosed and has at least one lower selection.

If, in addition, $X^*$ is strictly convex, there is a unique lower selection $T_0$ of T.

Proof: Let $\{(x_\alpha, w_\alpha), \alpha \in \Lambda\}$ be a net in T such that

$$x_\alpha \longrightarrow x(\Lambda),\ w_\alpha \rightharpoonup w(\Lambda),\ \sup\{\|w_\alpha\| \mid \alpha \in \Lambda\} < +\infty .$$

Let $(y,y^*) \in T$ be arbitrary. From the monotonicity of T, we have

$$\langle w_\alpha - y^*, x_\alpha - y \rangle \geqq 0 \text{ for all } \alpha \in \Lambda ,$$

and passing to a limit, we get $\langle w - y^*, x - y \rangle \geqq 0$. Since $(y,y^*) \in T$ was arbitrary, the maximal monotonicity of T gives $(x,w) \in T$. Thus the demiclosedness of T is proved and therefore T has at least one lower selection.

Further, let $X^*$ be strictly convex. Suppose that for some $x \in D(T)$, there are $w,z \in T(x)$ such that $\|w\| = \|z\| = c = \inf \{ \|x^*\| \mid x^* \in T(x) \}$. Then the convexity of $T(x)$ (see, e.g., [4]) gives $(w + z)/2 \in T(x)$, hence $\|(w + z)/2\| \geqq c$. But, on the other hand, $\|(w + z)/2\| \leqq c/2 + c/2 = c$. Thus the strict convexity of $X^*$ yields $w = z$. Hence, two different lower selections of T cannot exist. Q.E.D.

Let M be a nonempty subset of a normed linear space X. Following Kato [9], we introduce the set

$$\text{(2.1)} \qquad \operatorname{dint} M = \{ x \in M \mid \operatorname{cl} F_x(M) = X \} ,$$

where

$$\text{(2.2)} \qquad F_x(M) = \{ u \in X \mid \exists \{ t_n \} \subset R,\ t_n > 0,\ t_n \downarrow 0,\ \{ x + t_n u \} \subset M \} .$$

It should be noted that int M and the algebraic interior of M even are included in dint M.

<u>Example 2.1.</u> Let H be a separable Hilbert space, $\{e_i\}$ a total orthonormal system in H. We set

$$\text{(2.3)} \qquad M = \{ x = \sum_{i=1}^{\infty} t_i e_i \mid \{ t_i \} \subset R,\ \sum_{i=1}^{\infty} i \, | t_i | \leqq 1 \} .$$

It is easy to show that the set M is convex closed (hence, of the second category in itself) having empty algebraic interior, but dint $M \neq \emptyset$, even $M = \text{cl}\ (\text{dint}\ M)$.

<u>Lemma 2.2</u>. Let $T: X \longrightarrow 2^{X^*}$ be a monotone multivalued mapping from a normed linear space X to its dual $X^*$ and let $T_1$ be an arbitrary selection of T. Denote

$$SV(T) = \{ x \in D(T) \quad T(x) \text{ is a singleton} \},$$
$$MV(T) = D(T) \setminus SV(T).$$

Then, if $T_1$ is demicontinuous at $x \in \text{dint}\ D(T)$, the set $T(x)$ is a singleton:

$$(2.4) \quad C^d(T_1) \cap \text{dint}\ D(T) \subset SV(T), \text{ i.e.,}$$
$$MV(T) \cap \text{dint}\ D(T) \subset NC^d(T_1).$$

<u>Proof</u>: Let $x \in C^d(T_1) \cap \text{dint}\ D(T)$. Let w be an arbitrary element of the set $T(x)$. For every $u \in F_x(D(T))$ and the corresponding sequence $\{t_n\}$, $t_n > 0$, $t_n \downarrow 0$ (see (2.1) and (2.2)), from the monotonicity of T, we have

$$\langle T_1(x + t_n u) - w, (x + t_n u) - x \rangle \geqq 0, \ n = 1,2,\ldots,$$

and cancelling it by $t_n > 0$,

$$\langle T_1(x + t_n u) - w, u \rangle \geqq 0, \ n = 1,2,\ldots .$$

Using the demicontinuity (even the hemicontinuity only) of T, we then obtain that

$$\langle T_1(x) - w, u \rangle \geqq 0.$$

Since this inequality holds for each $u \in F_x(D(T))$, and $F_x(D(T))$ is a dense subset in X, it must be $T_1(x) = w$. But w was arbitrary element of the set $T(x)$, hence $T(x)$ is a singleton, i.e., $x \in SV(T)$. Thus the lemma is proved. Q.E.D.

Remark 2.1. If $T: X \longrightarrow 2^{X^*}$ is a maximal monotone multivalued mapping from a Banach space $X$ to $X^*$, with $\operatorname{int} D(T) \neq \emptyset$, (2.4) can be strengthened. The result of Rockafellar [18] says that $SV(T) \subset \operatorname{int} D(T)$ and that $T$ is locally bounded at any point of $\operatorname{int} D(T)$. From this and from (2.4), we can derive the following identity

$$C^d(T_1) \cap \operatorname{int} D(T) = SV(T).$$

Theorem 2.1. Let $X$ be a Banach space with a strictly convex dual $X^*$ and $T: X \longrightarrow 2^{X^*}$ a maximal monotone multivalued mapping.

Then the set

$$MV(T) \cap \operatorname{dint} D(T) = \{x \in \operatorname{dint} D(T) \mid T(x) \text{ is not a singleton}\}$$

is of the first category in $D(T)$.

If, moreover, $\operatorname{int} D(T) \neq \emptyset$, then the set

$$SV(T) \cap \operatorname{int} D(T) = \{x \in \operatorname{int} D(T) \mid T(x) \text{ is a singleton}\}$$

is dense residual in $\operatorname{int} D(T)$.

Proof: The first assertion follows immediately from Lemmas 2.2 and 1.3.

Further, let $\operatorname{int} D(T) \neq \emptyset$. Since the obvious inclusion $\operatorname{int} D(T) \subset \operatorname{dint} D(T)$ holds, the set $MV(T) \cap \operatorname{int} D(T)$ is of the first category in $D(T)$, hence also in $X$ and in the open nonempty set $\operatorname{int} D(T)$. Therefore the set

$$SV(T) \cap \operatorname{int} D(T) = \operatorname{int} D(T) \setminus (MV(T) \cap \operatorname{int} D(T))$$

is residual in int D(T) and, by Baire's category theorem, is dense in int D(T). Q.E.D.

Remark 2.2. Since $SV(T) \subset \text{int } D(T)$ (see [18]), we can write SV(T) instead of $SV(T) \cap \text{int } D(T)$ in Theorem 2.1.

Theorem 2.2. Let X be a Banach space with a dual $X^*$ which is strictly convex and has the property (H). Let $T: X \longrightarrow 2^{X^*}$ be a maximal monotone multivalued mapping.

Then:

(i) There exists a unique lower selection $T_0$ of T.

(ii) For each $x \in \text{dint } D(T)$ at which $T_0$ is continuous, T(x) is a singleton.

(iii) The set $C(T_0)$ of all those points at which $T_0$ is continuous, is residual $G_\delta$ in D(T), i.e., the set $NC(T_0) = D(T) \setminus C(T_0)$ is of the first category $F_\sigma$ in D(T).

(iv) If, in addition, $\text{int } D(T) \neq \emptyset$, the set $C(T_0) \cap \text{int } D(T)$ is dense residual $G_\delta$ in int D(T).

Proof: (i) is contained in Lemma 2.1.(ii) follows from Lemma 2.2 and the obvious inclusion $C(T_0) \subset C^d(T_0)$. (iii) is obtained by using (i) and Lemma 1.4. (iv) follows from (iii) and Baire's category theorem. Q.E.D.

Example 2.2. Let H be a separable Hilbert space, $\{e_i\}$ a total orthonormal system in H and $M \subset H$ the set defined by (2.3). Define the function $\varphi : H \longrightarrow R \cup \{+\infty\}$ as follows

$$\varphi(x) = 0, \text{ if } x \in M, \quad \varphi(x) = +\infty, \text{ if } x \notin M.$$

Obviously, $\varphi$ is a convex lower semicontinuous function. By [19], the subdifferential $\partial\varphi$ of $\varphi$ is a maximal monotone multivalued mapping from H to H, with $D(\partial\varphi) = M$. Hence, according to Example 2.1, int $D(\partial\varphi) = \emptyset$, but dint $D(\partial\varphi) \neq \emptyset$, and cl (dint $D(\partial\varphi)$) $= D(\partial\varphi)$ is of the second category in itself. It justifies the extension of our reasoning from the class of maximal monotone mappings $T$, with int $D(T) \neq \emptyset$, to that, with dint $D(T) \neq \emptyset$.

If int $D(T) \neq \emptyset$, then for the points $x \in C(T_0) \cap \text{int } D(T)$, we shall derive a little more still, namely, that at such points $x$, the mapping $T$ is (strongly) upper semicontinuous. We shall use the following lemma.

<u>Lemma 2.3</u>. Let $T: X \longrightarrow 2^{X^*}$ be a monotone multivalued mapping from a normed linear space $X$ to its dual $X^*$ such that int $D(T) \neq \emptyset$ and let $T_1$, $T_2$ be two arbitrary selections of $T$. Denote by $C(T_1)$, $C(T_2)$ the sets of all those points at which $T_1$, $T_2$ are continuous, respectively. Then

$$(2.5) \qquad C(T_1) \cap \text{int } D(T) = C(T_2) \cap \text{int } D(T).$$

<u>Proof</u>: In view of the symmetry of the conclusion, it suffices to prove the inclusion $\subset$ in (2.5). Let $x \in C(T_1) \cap$ $\cap$ int $D(T)$ be arbitrary. Recall that, by Lemma 2.2, $T_1(x) =$ $= T_2(x) = T(x)$. Let $\{x_n\} \subset D(T)$ be a sequence such that $x_n \longrightarrow x$. Since $x \in \text{int } D(T)$, we can suppose that $\{x_n\} \subset$ $\subset$ int $D(T)$. For each $n = 1,2,\ldots$, we find $v_n \in X$ so that

$$(2.6) \quad \|v_n\| \leqq 1 \text{ and } \|T_2(x_n) - T(x)\| - 1/n \leqq$$
$$\leqq \langle T_2(x_n) - T(x), v_n \rangle .$$

Further, for every $n = 1,2,\ldots$, we choose $t_n \in (0,1/n)$ so that $x_n + t_n v_n \in D(T)$.
The monotonicity of $T$ gives

$$\langle T_1(x_n + t_n v_n) - T_2(x_n),\ (x_n + t_n v_n) - x_n \rangle \geqq 0,$$

hence

$$\langle T_2(x_n), v_n \rangle \leqq \langle T_1(x_n + t_n v_n), v_n \rangle \quad ,$$

which together with (2.6) yields

$$\| T_2(x_n) - T(x) \| - 1/n \leqq \langle T_1(x_n + t_n v_n) - T(x), v_n \rangle \leqq$$
$$\leqq \| T_1(x_n + t_n v_n) - T(x) \| \quad .$$

But $x_n + t_n v_n \longrightarrow x$ and $x \in C(T_1)$. Therefore the last inequality gives that $\| T_2(x_n) - T(x) \| \longrightarrow 0$, i.e., $x \in C(T_2)$.

Q.E.D.

<u>Theorem 2.3</u>. Let X be a Banach space with a dual $X^*$ which is strictly convex and has the property (H). Let $T: X \longrightarrow 2^{X^*}$ be a maximal monotone multivalued mapping with $\operatorname{int} D(T) \neq \emptyset$.

Then the set of all those $x \in \operatorname{int} D(T)$ for which the set $T(x)$ is a singleton and T is upper semicontinuous at x, i.e., given $\varepsilon > 0$, there exists $\delta > 0$ such that for each $u \in D(T)$ fulfilling $\| x - u \| < \delta$, the set $T(u)$ is included in the $\varepsilon$-neighbourhood of $T(x)$, is dense residual $G_\delta$ in $\operatorname{int} D(T)$.

<u>Proof:</u> We set

$$C = \operatorname{int} D(T) \cap C(T_1),$$

where $T_1$ is an arbitrary selection of T. (Thanks to Lemma 2.3, the set C does not depend on the choice of $T_1$.) By Theorem 2.2 (iv), C is dense residual $G_\delta$ in int D(T). We shall show that C is that set of Theorem 2.3. Let $x \in \operatorname{int} D(T)$ be such that $T(x)$ is a singleton and $T$ is upper semicontinuous at $x$. Then we easily get $x \in C(T)$, hence $x \in C$. Conversely, let $x \in C$ be arbitrary. By Lemma 2.2, the set $T(x)$ is a singleton. We shall be proving that $T$ is upper semicontinuous at $x$. Let us suppose the contrary. Then there exists an $\varepsilon > 0$ and a sequence $\{(u_n, w_n)\} \subset T$ such that $u_n \longrightarrow x$ and

$$\| w_n - T(x) \| \geqq \varepsilon , \quad n = 1,2,\ldots . \tag{2.7}$$

We define the selection $T_2$ of $T$ as follows:

$$T_2(u_n) = w_n, \; n = 1,2,\ldots,$$

$$T_2(u) = \text{an arbitrary element of } T(u), \text{ for } u \notin \{u_n\} .$$

But since, by Lemma 2.3, $x \in C \subset C(T_2)$,

$$w_n = T_2(u_n) \longrightarrow T_2(x) = T(x),$$

which is in contradiction with (2.7). It means $T$ is upper semicontinuous at $x$. Q.E.D.

Remark 2.3. The second part of Theorem 2.1, and Theorem 2.3 are valid for arbitrary monotone multivalued mapping $T: X \longrightarrow 2^{X^*}$, with $\operatorname{int} D(T) \neq \emptyset$.

Remark 2.4. Let $T: X \longrightarrow 2^{X^*}$ be a maximal monotone multivalued mapping from a Banach space $X$ to its dual $X^*$ such that $\operatorname{int} D(T) \neq \emptyset$. Then, by Rockafellar's result [18], $D(T) \subset \operatorname{cl}(\operatorname{int} D(T))$, and hence, the set $D(T) \setminus \operatorname{int} D(T)$ is nowhere dense in $D(T)$. Therefore the text "in int D(T)" in Theorems 2.1 - 2.3 can be replaced by "in D(T)" (provided that $T$ is maximal monotone).

Remark 2.5. A somewhat different method for obtaining the results above, in the special case when X is reflexive, is given in [6].

Remark 2.6. Let $f: X \longrightarrow R \cup \{+\infty\}$ be a convex lower semicontinuous function, with $D(f) = X$ and int (dom $f) \neq \emptyset$. Then, it can be easily seen that the subdifferential $\partial f$ of f is a monotone multivalued mapping. Using [14], we immediately derive from Theorem 2.1 and Remark 2.3 that if $X^*$ is strictly convex, then the set of those points at which f is Gâteaux differentiable, is dense residual in int (dom f), which is included in Theorem 2 in [3]. It follows from Theorem 2.3 and Remark 2.3 by means of Proposition (ii) in [17] that if $X^*$ is strictly convex and has the property (H), then the set of those points at which f is Fréchet differentiable, is dense residual $G_\delta$ in int (dom f). This result is a little stronger than Theorem 1 in [3], where it is required for $X^*$ to be locally uniformly convex. However, our statement is included in [15].

Added in proof. After this paper had been prepared for publication, the author received the preprint by P. Kenderov and R. Robert: Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach, which will appear in C.R. Acad. Sci. Paris . Here it is independently shown that the conclusion of Theorem B is valid, if $X^*$ has the property (H), where nets are taken instead of sequences, without the assumption of strict convexity of $X^*$.

From the sketch of the proofs in the quoted work, it is obvious that our methods of the proofs are rather different.

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