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CONCERNING SPECTRAL CHARACTERIZATIONS OF THE RADICAL IN BANACH ALGEBRAS

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Abstract: An element r of a Banach algebra A belongs to the radical of A if and only if $|(1+q)r|_{\sigma} = 0$ for all q quasi-nilpotent in A.

Key words: Spectral radius, the radical of a Banach algebra.

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We consider an arbitrary Banach algebra A over the complex field. For x in A, let $\sigma(x)$ be the spectrum (taken in the unitization of A if A has no unit) and $|x|_{\sigma}$ the spectral radius of the element x. Denote by N the set of quasi-nilpotent elements in A, i.e. $N = \{x \in A: |x|_{\sigma} = 0\}$, and by rad A the (Jacobson) radical of A. It is well-known that $N \supset \mathrm{rad}\, A$, but this inclusion can often be proper. A characterization of algebras in which N = rad A is given in [1] (the set N is to be invariant under sums or, which is equivalent, under products). Thus although the radical is not - in general - simply the set of all quasi-nilpotents, it can nevertheless be characterized in terms of the spectral radius.

One such characterization [2] is based on the observation that $\sigma(a + r) = \sigma(a)$ for all $a \in A$, $r \in \text{rad } A$. We have shown in [2] that if, conversely, $\sigma(a + r) = \sigma(a)$ for all $a \in A$ and some $r \in A$, then it must be $r \in \text{rad } A$. In fact, the following theorem has appeared first in [2] although it was implicitly contained already in [1].

<u>Theorem 1</u>. Let A be a Banach algebra. Suppose $r \in A$ is such that $|a + r|_\sigma = 0$ for all $a \in N$. Then $r \in \text{rad } A$.

Another criterion has been known from early times of Banach algebras: if $r \in A$ is such that $|xr|_\sigma = 0$ for all $x \in A$, then $r \in \text{rad } A$. Now, Theorem 1 suggests that it should be possible to restrict the range of x's in this multiplicative criterion to some smaller subset of A being in some relation to the set N. We have remarked in [2] that it is not sufficient, for trivial reasons, to require the condition simply for all $x \in N$. However, it turns out that the appropriate restriction is to the elements of the form $x = 1 + a$ with $a \in N$. Indeed, the following result is a consequence of Theorem 1.

<u>Theorem 2</u>. Let A be a Banach algebra. Suppose $r \in A$ is such that $|(1 + a)r|_\sigma = 0$ for all $a \in N$. Then $r \in \text{rad } A$.

<u>Proof</u>. We show that $|a + r|_\sigma = 0$ for all $a \in N$; then the conclusion will follow by Theorem 1. Hence take an $a \in N$. It is enough to prove that, say, -1 does not belong to $\sigma(a + r)$. But we have the decomposition

$$1 + a + r = (1 + a)\{1 + [1 - (1 + a)^{-1}a]r\}$$

where the element

$$[1 - (1 + a)^{-1}a]r$$

is quasi-nilpotent by assumption. It follows that the element 1 + a + r, being represented as a product of two invertible elements, is invertible as well. This completes the proof.

We obtain similar corollaries as in [2]. Let us mention two of them.

Corollary 1. If R is a Banach space operator such that $|(1 + Q)R|_{\sigma} = 0$ for all Q quasi-nilpotent, then $R = 0$.

Corollary 2. The closed operator algebra generated by all the quasi-nilpotent operators on a Banach space is semi-simple.

R e f e r e n c e s

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