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A MATRIX INEQUALITY

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Abstract: Let $|A| = (A^* A)^{1/2}$ denote the Hermitian semidefinite component of the polar factorization of matrix A. A recently published paper established the inequality $\det(I + |A + B|) \leq \det(I + |A|) \det(I + |B|)$ for arbitrary matrices A and B; the proof uses techniques from Grassmann algebra. The objective of the present paper is to give a short direct proof of a matrix valued inequality having this determinantal inequality as an immediate consequence.

Key words: Matrix inequalities, singular values.

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Introduction: Let $|A| = (A^* A)^{1/2}$, as stated in the above abstract. The inequality

(1) $$\det(I + |A + B|) \leq \det(I + |A|) \det(I + |B|)$$

was established in a recent very interesting paper by Seiler and Simon [1]. These authors begin their paper by commenting that the triangle inequality

(2) $$|A + B| \leq |A| + |B|$$

is invalid (the inequality sign signifying that the right side minus the left side is positive semidefinite), and even that its consequence

(3) $$\det(|A + B|) \leq \det(|A| + |B|)$$

is invalid. Seiler and Simon then observe that the invalidity of (2) makes the valid inequality (1) of some interest. Indeed, were (2) valid, (1) would be suggested by the imprecise calculation $I + |A + B| \leq I + |A| + |B| \leq (I + |A|)(I + |B|)$, ignoring the technical difficulty of the non-Hermitian nature of the term $|A|\,|B|$ brought in by the last step. Although the proof of (1) given by Seiler and Simon is of considerable interest, particularly since it yields a number of additional results, it cannot be claimed to be elementary.

It is a not altogether evident fact that a modification of (2) does yield a valid matrix inequality, namely,

$$(4) \qquad |A + B| \leq U|A|U^* + V|B|V^*$$

for certain unitary matrices U and V (depending on A and B). This was recently established by Thompson, in [2]; in most applications (4) turns out to be every bit as satisfactory as the invalied inequality (2) would have been. Following the lead suggested by (4), it is natural to ask if the Seiler-Simon determinantal inequality (1) is a manifestation of an underlying matrix inequality, perhaps involving unitary matrices which cancel away upon taking determinants. The objective of this paper is to show that this indeed is the case. As a consequence, we obtain a proof of (1) involving only elementary ideas and nothing as complicated as Grassmann algebra.

2. Preliminary material. Let A and B be positive semidefinite Hermitian matrices. We shall use the following facts: If $P = I + A$, then each eigenvalue of P is at least 1; if $C = P^{-1/2}BP^{-1/2}$, then the eigenvalues $\gamma_1 \geq \dots \geq \gamma_n$ of C are

termwise dominated by the eigenvalues $\beta_1 \geq \dots \geq \beta_n$ of B; that is, $\gamma_i \leq \beta_i$ for $i = 1,2,\dots,n$. These are elementary facts. To prove the first, take v to be a unit eigenvector belonging to the smallest eigenvalue p_n of P and observe that $p_n = (Pv,v) = (v,v) + (Av,v) \geq (v,v) = 1$. To prove the second, let $f_1,\dots,f_n$ and $g_1,\dots,g_n$ be orthonormal eigenvectors of B and C, respectively, and take x to be a unit vector in the spans of $f_i,\dots,f_i$ and $P^{-1/2}g_1,\dots,P^{-1/2}g_i$. (These spans always have a nonzero intersection.) Then $(Bx,x) \leq \beta_i$, $x = P^{-1/2}y$ with y in the span of $g_1,\dots,g_i$, and $(Cy,y) \geq \gamma_i(y,y)$; also $(Px,x) \geq p_n$. Hence:

$$\beta_i \geq (Bx,x) = (P^{1/2}CP^{1/2}x,x) = (cy,y) \geq \gamma_i(y,y)$$
$$= \gamma_i(P^{1(2}x,P^{1/2}x) = \gamma_i(Px,x) \geq \gamma_i p_n \geq \gamma_i.$$

3. <u>The main result</u>. We shall prove the following theorem:

<u>Theorem:</u> Let A and B be $n \times n$ Hermitian matrices. Then unitary matrices U and V exist such that

$$\text{(5)} \qquad I + |A + B| \leq U(I + |A|)^{1/2}V(I + |B|)V^*(I + |A|)^{1/2}U^*.$$

Proof. First assume that A and B are positive semidefinite, so that $A = |A|$, $B = |B|$, $A + B = |A + B|$. Set $P = I + A = I + |A|$, and put $C = P^{-1/2}BP^{-1/2}$, as in the preceding section. Because the eigenvalues of C are termwise dominated by the eigenvalues of B, a unitary matrix W exists such that $C \leq WBW^*$. Indeed, if $C = W_1 \mathrm{diag}(\gamma_1,\dots,\gamma_n)W_1^*$, $B = W_2\mathrm{diag}(\beta_1,\dots,\beta_n)W_2^*$, where W_1, W_2 are unitary, take $W = W_1W_2^{-1}$. Thus

$$P^{-1/2}BP^{-1/2} \leq WBW^*,$$

yielding

$$B \leq P^{1/2}WBW^*P^{1/2},$$

$$P + B \leq P + P^{1/2}WBW^*P^{1/2},$$

$$P + B \leq P^{1/2}W(I + B)W^*P^{1/2},$$

$$I + A + B \leq (I + A)^{1/2}W(I + B)W^*(I + A)^{1/2}.$$

This completes the proof when A and B are positive semidefinite.

For the general case, we reason as follows, using (4) and the case already proved:

$$\begin{aligned} I + |A + B| &\leq I + U|A|U^* + V|B|V^* \\ &\leq (I + U|A|U^*)^{1/2}W(I + V|B|V^*)W^*(I + U|A|U^*)^{1/2} \\ &= U(I + |A|)^{1/2}(U^*WV)(I + |B|)(V^*W^*U)(I + |A|)^{1/2}U^*. \end{aligned}$$

The result is established upon renaming U^*WV as V.

From (5) it follows that the eigenvalues of $I + |A + B|$ are termwise dominated by the eigenvalues of $U(I + |A|)^{1/2}(V(I + |B|)V^*(I + |A|)^{1/2}U^*$. The determinant of the left hand side of (5) is therefore dominated by the determinant of the right hand side, yielding the inequality (1) of Seiler and Simon.

The following extension of (1) may be obtained: For arbitrary A, B,

$$|I + A + B| \leq U(I + |A|)^{1/2}V(I + |B|)V^*(I + |A|)^{1/2}U^*.$$

Indeed, by (4), $|I + A + B| \leq I + W|A + B|W^*$ for a certain unitary W; apply (5) to $WAW^* + WBW^*$.

R e f e r e n c e s

[1] E. SEILER and B. SIMON: An inequality among determinants, Proc. Nat.Acad. Sci. U.S.A. 72(1975), 3277-3278.

[2] R.C. THOMPSON: Convex and concave functions of singular values of matrix sums, submitted.

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