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Pavel Tomasta

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DECOMPOSITIONS OF COMPLETE k-UNIFORM HYPERGRAPHS INTO FACTORS WITH GIVEN DIAMETERS

Pavel TOMASTA, Bratislava

Abstract: The aim of this paper is to find an upper estimate for the minimal n (if it exists) with the property that K_n^k is decomposable into factors with given diameters. It will be shown that this property is hereditary.

Key words: Complete graphs, factor, diameter of graph.

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Introduction. The next considerations deal with k-uniform hypergraphs and give some generalizations of problems solved in [2] for graphs.

The purpose of this paper is to prove:

1. If a complete k-uniform hypergraph K_n^k with n vertices can be decomposed into m factors with given diameters (k,n,m are positive integers) then for any integer $N \geq n$ the hypergraph K_N^k can be also decomposed into m factors with the same diameters. This is a generalization of Theorem 1 of [2].

2. An upper estimate for the minimal n with the property that K_n^k is decomposable into factors with given diameters. This is an analogue of Theorem 4 of [2].

At first we give some definitions. A hypergraph is an ordered pair of sets $G = (V,H)$ where $H \subset P(V)$ (the potence of V). Let k be a positive integer. The hypergraph G is said to be a k-uniform hypergraph if for each $h \in H$ we have $|h| = k$. For $k = 2$ we obtain graphs. If the set H contains all the k-element subsets of V then G is said to be a complete k-uniform hypergraph and we denote G by K_n^k where $n = |V|$. The distance $d_G(x,y)$ of two vertices x and y in G is the length of the shortest path joining them. The diameter of the hypergraph G is defined by

$$d = \sup_{x,y \in V} d_G(x,y) .$$

The factor of G is a subhypergraph of G which contains all vertices of G. For unknown concepts see Berge [1].

<u>The general case.</u> Let $F^k(d_1,d_2,\dots,d_m) = t$ be the smallest integer (if it exists) such that the hypergraph K_t^k is decomposable into m factors with diameters $d_1,d_2,\dots,d_m$.

<u>Agreement:</u> We shall say that K_n^k is of type $T^k(d_1,d_2,\dots,d_m)$ if it is decomposable into m factors with diameters $d_1,d_2,\dots,d_m$.

The importance of the number $F^k(d_1,d_2,\dots,d_m)$ follows from

<u>Theorem 1:</u> Let m, n and $k \geq 2$ be positive integers and let $d_1,d_2,\dots,d_m$ be positive integers or symbols ∞. If $F^k(d_1,d_2,\dots,d_m) = n$ then for every integer $N \geq n$ the complete k-uniform hypergraph K_N^k can be decomposed into m factors with diameters $d_1,d_2,\dots,d_m$.

Remark: We denote the diameter of a disconnected hypergraph by the symbol ∞ .

First we prove the following

Lemma 1: Let $2 \leq k \leq n$ be integers. Then the hypergraph K_n^k cannot be decomposed into more than $\binom{n-2}{k-2}$ factors with diameter $d = 1$.

Proof of Lemma 1: Consider a factor of K_n^k with diameter one. Then every pair of its vertices belongs to at least one edge. There are $\binom{n}{2}$ pairs of vertices and every edge contains $\binom{k}{2}$ pairs of vertices. Consequently the number of edges of this factor is at least

$$\frac{\binom{n}{2}}{\binom{k}{2}} = \frac{n(n-1)}{k(k-1)} .$$

Thus for the number m_1 of the factors of K_n^k with diameter $d =$ $= 1$ we have

$$m_1 \quad \frac{\binom{n}{k}}{\frac{n(n-1)}{k(k-1)}} = \binom{n-2}{k-2} .$$

The proof is completed.

Proof of Theorem 1: The induction on N will be used. Suppose $d_1 \leq d_2 \leq \dots \leq d_m$.

1°. The first step of induction is evident: K_n^k is of type $T^k(d_1,d_2,\dots,d_m)$ by the assumption.

2°. Let $N \geq n$ and K_N^k be of type $T^k(d_1,d_2,\dots,d_m)$. Our aim is to prove that K_{N+1}^k is also of type $T^k(d_1,d_2,\dots,d_m)$. Denote its vertices by $1,2,\dots,N,v$. The hypergraph K_N^k with vertices $1,2,\dots,N$ is of type $T^k(d_1,d_2,\dots,d_m)$ by the induction hypothesis. Denote its factors with diameters $d_1,d_2,\dots$ $\dots,d_m$ by $F_1,F_2,\dots,F_m$.

We shall construct the factors $G_1,G_2,\dots,G_m$ of K^k_{N+1} as follows:

(a) If $h\in F_i$ then $h\in G_i$ for every $i = 1,2,\dots,m$.

(b) Let p be an arbitrary but fixed vertex of K^k_N and let $p_1,p_2,\dots,p_{k-1}$ be vertices of K^k_N different from p. Then the edge $\{v,p_1,p_2,\dots,p_{k-1}\}\in G_i$ if and only if $\{p,p_1,p_2,\dots$ $\dots,p_{k-1}\}\in F_i$, for every $i = 1,2,\dots,m$.

(c) If $d_1 = 1$ then there are exactly $\binom{N-1}{k-2}$ edges of type $\{v,p,q_1,q_2,\dots,q_{k-2}\}$ in the hypergraph K^k_{N-1} and by Lemma 1 we can give into every factor with diameter one at least one edge of this type. The remaining edges can be given into any factor with diameter one.

(d) Assume $d_1\geq 2$. Let q be some fixed vertex of K^k_N and $p\neq q$. If $\{p,q,q_1,q_2,\dots,q_{k-2}\}\in F_i$ then $\{p,v,q_1,\dots,q_{k-2}\}\in$ $\in G_i$, $i = 1,2,\dots,m$.

Now we prove that the factors $G_1,G_2,\dots,G_m$ have diameters $d_1,d_2,\dots,d_m$, respectively.

I. First we show that $d_i'\leq d_i$ for every $i = 1,2,\dots,m$ where d_i' is the diameter of G_i.

The edges $\{v,p_1,p_2,\dots,p_{k-1}\}\in G_i$ and $\{p,p_1,p_2,\dots$ $\dots,p_{k-1}\}\in F_i$ will be called "mutually corresponding". Analogously for the edges $\{v,p,q_1,q_2,\dots,q_{k-2}\}$ and $\{p,q,q_1,$ $q_2,\dots,q_{k-2}\}$. Further we say that the vertex x is "joined via p" with the vertex y if there exists an edge containing x and p and an edge containing y and p.

Let G_i be an arbitrary factor and x, y be an arbitrary pair of vertices of K^k_{N+1}. If $v \neq x,y$ then $d_{G_i}(x,y) \leq d_i$. Let now one of the vertices x, y be v. For example x = v. If $d_i = \infty$ then evidently $d_i' \leq d_i$. Thus it can be supposed $d_i < \infty$. We shall distinguish two cases.

1. $y \neq p$. Then there exists a chain connecting in F_i the vertices y and p. Take a shortest one. Let $\{p,p_1,p_2,\ldots,p_{k-1}\}$ be the last edge of this chain. Then from (b) it follows that the edge $\{v,p_1,p_2,\ldots,p_{k-1}\}$ belongs to G_i. Since $d_{G_i}(p_j,y) \leq d_i - 1$ for some $j = 1,2,\ldots,k-1$, we have $d_{G_i}(v,y) \leq d_i - 1 + 1 = d_i$.

2. $y = p$. If $d_i = 1$ then $d_{G_i}(v,y) = 1$, because some edge of type $\{v,p,q_1,q_2,\ldots q_{k-2}\}$ belongs to G_i (it follows from (c)).

If $d_i > 1$ then $d_{G_i}(v,y) \leq 2$. Thus $d_i' \leq d_i$ and we proved the first part.

II. We shall show that $d_i \leq d_i'$. Let $d_i = \infty$ and $d_i' < \infty$. Then we can find two vertices $x,y \neq v$ such that there exists a chain $x,h_1,x_1,h_2,\ldots,h_t,x_t = y$ in G_i but no chain joining x and y in F_i. Every edge $h_r \notin F_i$ in this chain can be replaced by the "mutually corresponding" edge in F_i ensuring the joining between x and y. This is a contradiction to the assumption that there is no chain between x and y in F_i.

Let now $1 < d_i < \infty$. Then there exist two vertices x and y with the property $d_{F_i}(x,y) = d_i$. Let x', y' be the vertices from a shortest chain between x and y in G_i which are "joined via v" and either $d_{G_i}(x',y') = 2$ or $d_{G_i}(x',y') = 1$.

Then x' and y' are "joined via p" by the chain of length either 2, if $d_{G_i}(x',y') = 2$, or 1 if $d_{G_i}(x',y') = 1$ with "mutually corresponding" edges in F_i. Consequently, $d_{G_i}(x,y) = d_i$.

Let now $d_i = 1$. From the case I we have $d_i' \leq d_i$ what implies $d_i' = d_i = 1$.

Since $d_i \leq d_i'$ and $d_i' \leq d_i$ we have $d_i = d_i'$ for every $i = 1,2,\dots,m$ and this completes the proof.

Corollary 1: Let $F_1,F_2,\dots,F_m$ be factors of a decomposition of K_t^k with diameters $d_1,d_2,\dots,d_m$, respectively. Then there exists a decomposition of K_{t+1}^k into factors $G_1,G_2,\dots,G_m$ with diameters $d_1,d_2,\dots,d_m$ such that $F_i \subset G_i$, $1 \leq i \leq m$.

Proof: It is evident.

Decompositions with the diameter one. Theorem 1 does not ensure the existence of the number $F^k(d_1,d_2,\dots,d_m)$. Our aim in this section is to ensure it in the case that at least one of the diameters is one.

Lemma 2: Let $k \geq 3$ be an integer. Then

$$F^k(1,1) = k + 1 \text{ if } k \geq 5 \text{ and}$$

$$F^k(1,1) = k + 2 \text{ if } k = 3,\ 4.$$

Proof: If $k = 3$ then we consider K_5^3. Let G_1 contain the edges $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$ and G_2 contain all the remaining edges. Then evidently both of them have diameters equal to one.

We show that $F^3(1,1) > 4$. Consider a decomposition of K_4^3 into two factors G_1 and G_2 with diameters equal to one. Since G_1 has the diameter one it must contain at least three edges.

Hence G_2 contains only one edge. Thus $d_{G_2} = \infty$ and $F^3(1,1) > 4$.

If $k = 4$ then let $G_1 = \{\{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}\}$ and G_2 be its complement. Evidently $d_{G_1} = d_{G_2} = 1$.

Now let $F^4(1,1) \leq 5$. Then one of the factors G_1 and G_2 contains two or less edges. Hence it cannot have the diameter one.

If $k \geq 5$ then let $G_1 = \{\{1,2,\ldots,k\}, \{1,2,\ldots,k-1,k+1\}, \{2,3,\ldots,k+1\}\}$ and G_2 be its complement. The factors G_1, G_2 have the diameters equal to one and the proof is finished.

It will be said that a decomposition R of K_t^k has the property (P) if each factor of R covers all vertices of K_t^k.

We shall prove some trivial but useful statements for our further considerations.

Lemma 3: Let $n \geq 1$ be integer and $t = 5.2^{n-1}$. Then there exists a decomposition of K_t^2 into 2^n factors with property (P).

Proof: If $n = 1$ then $t = 5$ and K_5^2 can be evidently decomposed into two factors with the property (P). If $n \geq 2$ then t is even and there exists a decomposition of K_t^2 into $5.2^{n-1} - 1$ 1-factors. Since $5.2^{n-1} - 1 > 2^n$ the proof is finished.

Corollary 2: Let $n \geq 1$, $k \geq 2$ be integers and $t = (k+2)2^{n-1}$. Then there exists a decomposition with property (P) of K_t^{k-1} into 2^n factors.

The proof follows immediately.

Theorem 2: Let m and $k \geq 3$ be integers. Then $F_m^k(1)$ exists and

$$F_m^k(1) \leq (k+2)2^{\{\log_2 m\}-1} .$$

Proof: We shall show that $F^k_{2^n}(1)$ exists and $F^k_{2^n}(1) \leq$ $\leq (k+2)2^{n-1}$ for every integers $n \geq 1$ and $k \geq 3$. The induction on n will be used.

1^o. Let $n = 1$. Then from Lemma 2

$$F^k_2(1) \leq k + 2 = (k+2)2^0.$$

2^o. Suppose $F^k_{2^n}(1) \leq (k+2)2^{n-1}$. Put $t = (k+2)2^{n-1}$ and consider K^k_{2t} with the vertex set $V = V_1 \cup V_2 = \{1_1, 2_1, \ldots, t_1\} \cup \{1_2, 2_2, \ldots, t_2\}$. Let $\{T^1_1, T^1_2, \ldots, T^1_{2n}\}$ and $\{T^2_1, T^2_2, \ldots, T^2_{2^n}\}$ be decomposition with property (P) of the hypergraph K^{k-1}_t with the vertex set V_1 and V_2, respectively. Such decompositions are warranted by Corollary 2.

Let $\alpha_1 = (1_1, 2_1, \ldots, (2^n)_1)$ and $\alpha_2 = (1_2, 2_2, \ldots, (2^n)_2)$ be permutations. By the induction assumption the hypergraph K^k_t with the vertex set V_1 and V_2, respectively can be decomposed into 2^n factors with diameters equal to one. Denote these factors by F^1_j and F^2_j, $j = 1, 2, \ldots, 2^n$.

Now we shall construct the decomposition of K^k_{2t} into 2^{n+1} factors with diameters equal to one.

(1) If $\{(v_1)_1, (v_2)_2, \ldots, (v_{k-1})_1\} \in T^1_i$ then

$$\{(v_1)_1, (v_2)_1, \ldots, (v_{k-1})_1, (\alpha^j_2(i))_2\} \in G^1_j.$$

If $\{(v_1)_2, (v_2)_2, \ldots, (v_{k-1})_2\} \in T^2_i$ then

$$\{(v_1)_2, (v_2)_2, \ldots, (v_{k-1})_2, (\alpha^j_1(i))_1\} \in G^2_j$$

for every $1 \leq i \leq 2^n$, $1 \leq j \leq 2^n$.

(2) If $\{(v_1)_1, (v_2)_1, \dots, (v_{k-1})_1\} \in T_i^1$ then

$$\{(v_1)_1, (v_2)_1, \dots, (v_{k-1})_1, s_2\} \in G_i^1.$$

If $\{(v_1)_2, (v_2)_2, \dots, (v_{k-1})_2\} \in T_i^2$ then

$$\{(v_1)_2, (v_2)_2, \dots, (v_{k-1})_2, s_1\} \in G_i^2$$

for every $2^n < s \leq t$, $1 \leq i \leq 2^n$.

(3) If $h \in F_j^2$ then $h \in G_j^1$ and

if $h \in F_j^1$ then $h \in G_j^2$ for every $1 \leq j \leq 2^n$.

(4) All the remaining edges can be added into an arbitrary factor.

Now it will be verified that the diameters of the factors G_j^1 and G_j^2 are equal to one. Let G_j^1 be an arbitrary of them.

(i) If $a_1, b_1 \in V_1$ then there exists i such that

$\{a_1, b_1, (x_1)_1, (x_2)_1, \dots, (x_{k-3})_1\} \in T_i^1$ for some $x_1, x_2, \dots$

$\dots, x_{k-3}$. Since (1) holds we have for $s = \alpha_2^j(i)$:

$$\{a_1, b_1, (x_1)_1, (x_2)_1, \dots, (x_{k-3})_1, s_2\} \in G_j^1.$$

Thus the distance between a_1 and b_1 is equal to one.

(ii) If $a_1 \in V_1$ and $b_2 \in V_2$ then two cases are possible.

1. $1 \leq b \leq 2^n$. Then there exists i, $\alpha_2^j(i) = b$. From the property (P) it follows that there exists

$\{a_1, (x_1)_1, (x_2)_1, \dots, (x_{k-2})_1\} \in T_i^1$ for some $x_1, x_2, \dots$

$\dots, x_{k-2}$. Since (1) holds we have

$$\{a_1,(x_1)_1,(x_2)_1,\dots,(x_{k-2})_1,b_2\} \in G_j^1.$$

Thus the distance between a_1 and b_2 is equal to one.

2. $2^n < b \leq t$. From the property (P) we have that there exists

$$\{a_1(x_1)_1,(x_2)_1,\dots,(x_{k-2})_1\} \in T_j^1 \text{ for some } x_1,x_2,\dots,x_{k-2}.$$

Since (2) is true we obtain

$$\{a_1,(x_1)_1,(x_2)_1,\dots,(x_{k-2})_1,b_2\} \in G_j^1.$$

Thus the distance between a_1 and b_2 is equal to one.

(iii) If $a_2,b_2 \in V_2$ then there exists in F_j^2 some edge which contains both of these vertices. Since (3) holds this edge is also in G_j^1. Thus the distance between a_2 and b_2 is equal to one.

The verification for the factors G_j^2 can be made analogously.

We showed that $F^k_{2^{n+1}}(1) \leq (k+2)2^n$ and the induction is completed. Put $q = \{\log_2 m\}$. Since $F_m^k(1) \leq F_q^k(1)$ the proof is finished.

Lemma 4: Let $m \geq 2$, $k \geq 3$ and $1 \leq d_1,d_2,\dots,d_m$ be integers. If $F^k(d_1,d_2,\dots,d_m,1)$ exists then

$$F^k(d_1+1,d_2,d_3,\dots,d_m,1) \leq F^k(d_1,d_2,\dots,d_m,1) + k - 1.$$

Proof: Put $t = F^k(d_1,d_2,\dots,d_m,1)$ and consider a decomposition of K_t^k with the vertex set $\{v_1,v_2,\dots,v_t\}$ into factors $F_1,F_2,\dots,F_{m+1}$ with diameters equal to $d_1,d_2,\dots,d_m$, , respectively. Add the vertices $y_1,y_2,\dots,y_{k-1}$ to K_t^k.

Now we shall construct a decomposition of K^k_{t+k-1} into factors $G_1,G_2,\dots,G_{m+1}$ with diameters $d_1+1,d_2,d_3,\dots,d_m,1$, respectively.

Since the diameter of F_1 is d_1 there exist two vertices v_p and v_q with $d_{F_1}(v_p,v_q) = d_1$. By Theorem 1 there exists a decomposition of K^k_{t+k-1} into factors $H_1,H_2,\dots,H_{m+1}$ with diameters $d_1,d_2,\dots,d_m,1$, respectively. Consider accurately this decomposition. Moreover, using Corollary 1 we have $F_i \subset H_i$ for every $i = 1,2,\dots,m+1$.

Now put $G_x = H_x$ for every $2 \leq x \leq m$ except such $x_0 \neq 1$ for which $\{v_p,y_1,y_2,\dots,y_{k-1}\} \in H_{x_0}$.

Let G_1 contain the factor F_1 and the edge $\{v_p,y_1,y_2,\dots$ $\dots,y_{k-1}\}$. Let G_{m+1} contain the factor H_{m+1} and the edges from $H_1 - F_1$. Let $G_{x_0} = H_{x_0} - \{v_p,y_1,y_2,\dots,y_{k-1}\}$.
The diameter of G_1 is equal to d_1+1, because $d_{G_1}(y_{k-1},v_q) =$ $= d_1 + 1$.

It is easy to see that the factors $G_1,G_2,\dots,G_{m+1}$ form the required decomposition of K^k_{t+k-1} and this completes the proof.

<u>Theorem 3:</u> Let $m,k \geq 3$, $1 \leq d_1,d_2,\dots,d_m$ be integers and at least one $d_i = 1$. Then

$$\text{(N)} \qquad F^k(d_1,d_2,\dots,d_m) \leq F^k_m(1) + (k-1)\sum_{j=1}^{m}(d_j - 1).$$

Proof: From Theorem 2 it follows that $F^k_m(1)$ exists. Then by Lemma 4 $F^k(d_1,d_2,\dots,d_m)$ exists, too. The inequality (N) follows immediately from Lemma 4 and the proof is completed.

The upper estimate of the number $F^k(d_1,d_2,\dots,d_m)$ can be improved for some values of parameters $d_1,d_2,\dots,d_m,m$.

Theorem 4: Let $k\geq 3$, $q\geq 3$, $q<m$, $2\leq d_1\leq\dots\leq d_q$ be integers and $d_{q+1} = d_{q+2} = \dots = d_m = 1$. Then

$$F^k(d_1,d_2,\dots,d_m)\leq \max\{F^2(d_1,d_2,\dots,d_q),\ F^k_{m-q}(1),m-q\}+$$
$$+\max\{(k-2)d_q,\ 3(m-q)\}.$$

Proof: Put $m_1 = \max\{F^2(d_1,d_2,\dots,d_q),\ F^k_{m-q}(1),m-q\}$

and $m_2 = \max\{(k-2)d_q,\ 3(m-q)\}$.

Let M_1 and M_2 be sets of cardinality m_1 and m_2, respectively. Lemma 2 of [3] implies that there exists a decomposition of $K^2_{m_1}$ with the vertex set M_1 into factors $F_1,F_2,\dots,F_q$ with diameters $d_1,d_2,\dots,d_q$. Now we shall construct the factors $G_1,G_2,\dots,G_q$ of the hypergraph $K^k_{m_1+m_2}$ with diameters $d_1,d_2,\dots,d_q$.

Choose from M_2 any $(k-2)d_r$ vertices v_i^j, $1\leq j\leq k-2$, $1\leq i\leq d_r$, $1\leq r\leq q$. Let x_r and y_r be vertices of M_1 such that $d_{F_r}(x_r,y_r) = d_r$.

1. If the edge $\{a,b\}\in F_r$ and if $d_{F_r}(x_r,n) = d_{F_r}(x_r,a) = d$ then $\{a,b,v_d^1,v_d^2,\dots,v_d^{k-2}\}\in G_r$.

2. If the edge $\{a,b\}\in F_r$ and if $d_{F_r}(x,b) = d_{F_r}(x_r,a) + 1 = d$ then $\{a,b,v_d^1,v_d^2,\dots,v_d^{k-2}\}\in G_r$.

3. If $\{x_o,y_o\}\in F_r$ is some fixed edge and if

$M_3 = M_2 - \{v_i^j \mid 1\leq j\leq k-2,\ 1\leq i\leq d_r\}$

has cardinality $|M_3| \geq k - 2$ then

$\{x_0, y_0, v_1, v_2, \ldots, v_{k-2}\} \in G_r$ for every $(k - 2)$-tuple $\{v_1, \ldots, v_{k-2}\} \in M_3$.

If $|M_3| = s < k - 2$ then

$\{x_0, y_0, v_1, v_2, \ldots, v_s, v_1^1, v_1^2, \ldots, v_1^{k-2-s}\} \in G_r$ where $\{v_1, \ldots, v_s\} = M_3$.

It is easy to see that the diameter of G_r is equal to d_r. For example $d_{G_r}(v_1^1, v_{d_r}^1) = d_r$.

Now we shall construct the factors $G_{q+1}, G_{q+2}, \ldots, G_m$. Since $F_{m-q}^k(1) \leq m_1$ there exist the factors $F_{q+1}, F_{q+2}, \ldots, F_m$ of $K_{m_1}^k$ (on the vertex set M_1) with diameters equal to one. Let $\{T_{q+1}, T_{q+2}, \ldots, T_m\}$ be a decomposition with the property (P) of the hypergraph $K_{m_2}^{k-1}$ with the vertex set M_2. Such a decomposition exists from Lemma 3.

Let us have $q + 1 \leq r \leq m$.

1. If $h \in F_r$ then $h \in G_r$.

2. Let α be a permutation on vertices $p_1, \ldots, p_{m-q} \in M_1$ with $\alpha(p_1) = p_2$, $\alpha(p_2) = p_3, \ldots, \alpha(p_{m-q}) = p_1$.

If $\{y_1, y_2, \ldots, y_{k-1}\} \in T_i$ then $\{y_1, y_2, \ldots, y_{k-1}, \alpha^r(p_i)\} \in G_r$.

3. If $\{y_1, y_2, \ldots, y_{k-1}\} \in T_r$ then $\{y_1, y_2, \ldots, y_{k-1}, x\} \in$ $\in G_r$, where $x \in M_1 - \{p_1, p_2, \ldots, p_{m-q}\}$.

The remaining edges of $K_{m_1+m_2}^k$ can be inserted into an arbitrary factor with diameter one.

The factors $G_1, G_2, \ldots, G_m$ evidently form the required decomposition of $K^k_{m_1+m_2}$ and this completes the proof.

<u>The case</u> $m = 2$. In this section there is obtained a complete solution of the problem of decomposing complete k-uniform hypergraphs into two factors with given diameters.

<u>Lemma 5:</u> Let G be a k-uniform hypergraph with diameter $d \geq 2$. Then its complement $\overline{G}$ has the diameter

$$d_{\overline{G}} \leq 2 \text{ if } k = 3 \quad \text{and}$$

$$d_{\overline{G}} = 1 \text{ if } k \geq 4 \text{ .}$$

Proof: Let x_o and y_o be vertices of G such that $d_G(x_o,y_o) \geq 2$. All the edges containing x_o and y_o belong to $\overline{G}$. Let x and y be arbitrary vertices of $\overline{G}$. If $k \geq 4$ then there exists an edge in $\overline{G}$ containing x_o,y_o,x,y. Thus $d_{\overline{G}}(x,y) = 1$. If $k = 3$ then $\{x_o,y,y_o\} \in \overline{G}$ and $\{x_o,x,y_o\} \in \overline{G}$. Hence $d_{\overline{G}}(x,y) \leq$ ≤ 2 and the proof is finished.

<u>Lemma 6:</u> Let G be a 3-uniform hypergraph with diameter $d \geq 3$. Then its complement $\overline{G}$ has diameter equal to one.

Proof: Let x_o,y_o be vertices of G such that $d_G(x_o,y_o) \geq$ ≥ 3. Then let x, y be any pair of vertices in G. There evidently exists a vertex z_o such that $\{x,y,z_o\} \notin G$. Hence $\{x,y,z_o\} \in \overline{G}$ and this completes the proof.

These lemmas imply the following results:

<u>Theorem 5</u>:

1. If $d_1 = 1$ and $d_2 = \infty$, then $F^k(d_1,d_2) = k$.
2. If $d_1 = 1$ and $d_2 = 1$, then $F^k(d_1,d_2) = k + 1$ if $k \geq 5$,
   $F^k(d_1,d_2) = k + 2$ if $k = 3,4$.

3. If $d_1 = 1$ and $d_2 = 2$, then $F^k(d_1,d_2) = k + 1$ if $k \geq 4$,

$F^k(d_1,d_2) = 5$ if $k = 3$.

4. If $d_1 = 2$ and $d_2 = 2$, then $F^k(d_1,d_2)$ does not exist if $k \geq 4$,

$F^k(d_1,d_2) = 4$ if $k = 3$.

5. If $d_1 \geq 2$ and $d_2 \geq 3$, then $F^k(d_1,d_2)$ does not exist.

6. If $d_1 = 1$ and $3 \leq d_2 < \infty$, then

$$F^k(d_1,d_2) = \frac{kd_2}{2} + 1 \text{ if } d_2 \text{ is even,}$$

$$F^k(d_1,d_2) = \frac{k(d_2 + 1)}{2} \text{ if } d_2 \text{ is odd.}$$

Proof: We shall denote the vertices by naturals and the factors of a decomposition by G_1 and G_2.

1. G_1 contains $\{1,2,\dots,k\}$ and G_2 is empty.

2. If follows from Lemma 2.

3. If $k \geq 4$, then it follows from Lemma 5.

If $k = 3$, then $G_1 = \{\{1,2,3\}, \{1,2,4\}, \{1,3,5\}, \{2,4,5\}, \{3,4,5\}, \{1,2,5\}\}$. Put $G_2 = \overline{G}_1$.

4. If $k \geq 4$, then it follows from Lemma 5.

If $k = 3$, then $G_1 = \{\{1,2,4\}, \{1,3,4\}\}$ and $G_2 = \overline{G}_1$.

5. It follows from Lemmas 5 and 6.

6. It directly follows from the construction of a chain of length equal to d_2.

It remains to prove the existence of the number

$F^k(d_1,\dots,d_m)$ for arbitrary $d_1,\dots,d_m$ and to give an upper estimate for this.

This problem is partially solved in [4] for the case $m > k$. In [5] it is proved that if $m \le k$ and 3 $d_1, d_2, \dots, d_m$ then such a number does not exist.

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Matematický ústav SAV

Obrancov mieru 49

Bratislava

Československo

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