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ON L^p-ESTIMATES FOR SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

Miroslav KRBEC, Praha

Abstract: The purpose of this paper is to generalize the known regularity results concerning the Dirichlet problem for linear elliptic partial differential equations of order 2m with L^∞-coefficients to the case of general boundary value problem in variational formulation. A regularity theorem in Sobolev spaces $W_p^m(\Omega)$ is proved for p near to 2.

Key words: Sobolev spaces, complementary conditions, properly elliptic operators, a priori estimates, regularity.

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1. Introduction. In the papers [3],[5], the authors obtained regularity theorems for weak solutions of the Dirichlet problem for linear elliptic partial differential divergence equations with bounded and measurable coefficients. These results are an extension of the known classical existence and unicity theorems in Sobolev spaces $W_2^m(\Omega)$, where Ω is a bounded domain in N-space E^N, on spaces $W_p^m(\Omega)$, where p is near to 2 (m = 1 in [3] and m arbitrary integer in [5]). For p large enough there are counterexamples (see, e.g.[3]). In this paper there will be proved a regularity theorem of the mentioned type for general boun-

dary value problem in variational form. At first there is obtained a suitable a priori estimate for functions in $W_p^m(\Omega)$ which represents certain bounded linear forms on $W_{p'}^m(\Omega)$ ($p \geq 2$ and $p' = p/(p-1)$) - "right hand side of equations". These estimates are then applied to general b.v.p. according to the method used in [5]. The precise statement of the problem and the main result are in Section 3.

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Notation. The symbol Ω means in all this paper a bounded domain in E^N ($N \geq 2$) with regular boundary $\partial\Omega$ in the sense of [2]. Points of E^N will be denoted by $x = (x_1, \ldots, x_N)$. If $\alpha = (\alpha_1, \ldots, \alpha_N)$, then the operator D^α is defined in a usual way, i.e.

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}},$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_N$ is the length of α.

All the following functional spaces are real. Let $p > 1$, $s \geq 1$ and set $p' = p/(p-1)$. Let us introduce in $C^\infty(\overline{\Omega})$ the norm

$$\|u\|_{s,p} = \Big(\int_\Omega \sum_{|\alpha| \leq s} |D^\alpha u|^p \, dx \Big)^{1/p} .$$

The completion of $C^\infty(\overline{\Omega})$ with respect to this norm is the Sobolev space $W_p^s(\Omega)$. The space $W_p^{s-1/p}(\partial\Omega)$, resp. $W_p^{-s-1/p}(\partial\Omega)$ is defined as a completion of $C^\infty(\partial\Omega)$ with respect to the norm

$$\|u\|_{s-1/p,p} = \inf_{\substack{v \in C^\infty(\overline{\Omega}) \\ v = u \text{ on } \partial\Omega}} \|v\|_{s,p} ,$$

resp.

$$\|u\|_{-s-1/p,p} = \sup_{\substack{v \in C^\infty(\overline{\Omega}) \\ \|v\|_{s+1/p,p'} = 1}} \int_{\partial\Omega} u v \, dS \ .$$

The norms in spaces $L^p(\Omega)$ and $L^p(\partial\Omega)$ will be denoted by $\|\cdot\|_{0,p}$.

The word "operator" means "linear partial differential operator". Different constants are sometimes denoted by the same symbol.

2. <u>A priori estimates.</u> Let

$$A = \sum_{|\alpha|, |\beta| \le m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta) \ ,$$

where $a_{\alpha\beta} \in C^\infty(\overline{\Omega})$ and $m \ge 1$ is a fixed integer. Denote by $P(x, .)$ the characteristic polynomial of A. Suppose that A is elliptic in $\overline{\Omega}$. Then A is properly elliptic in $\overline{\Omega}$ (see, e.g. [2]), i.e. for each $x \in \overline{\Omega}$ and linearly independent vectors ξ , $\xi' \in E^N$, the equation $P(x, \xi + t\xi') = 0$, where t is complex variable, has no real roots and just half of its roots (including multiplicity) has positive imaginary part. Denote these roots by $t_1(x, \xi, \xi'), \ldots$
$\ldots, t_m(x, \xi, \xi')$ and define the polynomial

$$M(x, \xi, \xi', t) = (t - t_1(x, \xi, \xi')) \cdot \ldots \cdot (t - t_m(x, \xi, \xi')).$$

<u>Definition 2.1.</u> Any finite set of operators on $\partial\Omega$ is called a normal set on $\partial\Omega$ if orders of these operators are different and $\partial\Omega$ is non-characteristic for each of them.

Any system of k operators on $\partial\Omega$ is said to be a Dirichlet set of order k if it is a normal set on $\partial\Omega$ and orders of these operators are less than k.

Definition 2.2. Let $\{H_j\}_{j=0}^{k-1}$ be any system of operators on $\partial\Omega$ with characteristic polynomials $\{Q_j(x, .)\}_{j=0}^{k-1}$. The system $\{H_j\}_{j=0}^{k-1}$ is said to satisfy the complementary conditions with respect to A on $\partial\Omega$ (or to cover A on $\partial\Omega$) if for each $x \in \partial\Omega$ and $\xi, \xi' \in E^N \setminus \{0\}$ such that ξ is tangent and ξ' is normal to $\partial\Omega$ at the point x, the polynomials $\{Q_j(x, \xi + t\xi')\}_{j=0}^{k-1}$ in the complex variable t are linearly independent mod the polynomial $M(x, \xi, \xi', t)$.

Let $\{B_j\}_{j=0}^{2m-1}$ be a Dirichlet set of order 2m on $\partial\Omega$. Suppose that coefficients of B_j are in $C^\infty(\partial\Omega)$ and denote by m_j the order of B_j. Then there exists a unique system $\{B_j'\}_{j=0}^{2m-1}$ of operators on $\partial\Omega$ which is a Dirichlet one of order 2m on $\partial\Omega$ so that coefficients of B_j' are in $C^\infty(\partial\Omega)$, the order of B_j' is $2m - 1 - m_j$ and the equality

$$\int_\Omega v A u \, dx = \int_\Omega u A' v \, dx + \int_{\partial\Omega} \sum_{j=0}^{2m-1} B_j u \, B_j' v \, dS ,$$

where A' is the formal adjoint of A, holds for each $u, v \in C^\infty(\bar\Omega)$ (see [2]). Denote

$$U = \{v \in C^\infty(\bar\Omega);\ B_j v = 0 \text{ on } \partial\Omega,\ 0 \le j \le m-1\},$$

$$U' = \{v \in C^\infty(\bar\Omega);\ B_j' v = 0 \text{ on } \partial\Omega,\ m \le j \le 2m-1\}.$$

Theorem 2.1. Let $A^{-1}(0) \cap U = \{0\}$ and suppose that $\{B_j\}_{j=0}^{m-1}$ satisfy the complementary conditions with respect to A on $\partial\Omega$. Then for each $p > 1$ there exists $c_p > 0$ so that the inequality

$$\|u\|_{m,p} \le c_p \Big(\sup_{\substack{v \in U' \\ \|v\|_{m,p'} = 1}} \int_\Omega v\, A u \, dx + \sum_{j=0}^{m-1} \|B_j u\|_{m-m_j-1/p,p} \Big)$$

holds for each $u \in C^\infty(\overline{\Omega})$.

Proof of this theorem can be found in [6] .

Remark 2.1. Under assumptions of Theorem 2.1 for each $p > 1$ there is $c_p > 0$ such that the inequality

$$(2.1) \quad \|u\|_{m,p} \le c_p \Big(\sup_{\substack{v \in V' \\ \|v\|_{m,p'} = 1}} \int_\Omega v\, A u \, dx + \sum_{j=0}^{m-1} \|B_j u\|_{m-m_j-1/p,p} \Big)$$

where

$$V' = \{ v \in C^\infty(\overline{\Omega});\ B'_j v = 0 \text{ on } \partial\Omega \text{ for } j \text{ such that } m \le j \le 2m-1,\ m_j \ge m \},$$

holds for each $u \in C^\infty(\overline{\Omega})$.

Proof. Obviously $U' \subset V'$. Let $U'_{p'}$, resp. $V'_{p'}$, be the closure of U', resp. V', in $W^m_{p'}(\Omega)$ and $v \in V'_{p'} \cap C^\infty(\overline{\Omega})$. Then there is a function $w \in C^\infty(\overline{\Omega})$ such that w belongs to the closure of $\mathcal{D}(\Omega)$ in $W^m_{p'}(\Omega)$ and $v - w \in U'$ (see, e.g. [7]). As $\mathcal{D}(\Omega) \subset U$, we have $v = (v - w) + w \in U'_{p'}$. Thus $V'_{p'} \subset U'_{p'}$ and (2.1) follows.

3. A regularity theorem. At first let us fix notation. Let

$$A = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta) ,$$

$$\tilde{A} = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D^\alpha (\delta_{\alpha\beta} D^\beta) ,$$

where $a_{\alpha\beta} \in L^\infty(\Omega)$ and $\delta_{\alpha\beta}$ is the Kronecker symbol. Denote by B and $\tilde{B}$ the corresponding bilinear forms on

$W_p^m(\Omega) \times W_{p'}^m(\Omega)$, i.e.

$$B(u,v) = \int_\Omega \sum_{|\alpha|,|\beta| \le m} a_{\alpha\beta} D^\alpha v \, D^\beta u \, dx ,$$

$$\tilde{B}(u,v) = \int_\Omega \sum_{|\alpha|,|\beta| \le m} \delta_{\alpha\beta} D^\alpha v \, D^\beta u \, dx .$$

Let $\{B_j\}_{j=0}^{m-1}$ be a Dirichlet set of order m on $\partial\Omega$ of operators with coefficients in $C^\infty(\partial\Omega)$. Denote by m_j the order of B_j. Let us fix some integer $r \in \langle 0, m-1 \rangle$ and set

$$V = \{v \in C^\infty(\bar{\Omega});\ B_j v = 0 \text{ on } \partial\Omega \ ,\ 0 \le j \le r-1\} .$$

By V_p denote the closure of V in $W_p^m(\Omega)$. There exists a unique normal set on $\partial\Omega$ of operators $\{F_j\}_{j=0}^{m-1}$ with coefficients in $C^\infty(\partial\Omega)$ so that the order of F_j is $2m - 1 - m_j$ and the equality

$$\tilde{B}(u,v) = \int_\Omega v \tilde{A} u \, dx + \int_{\partial\Omega} \sum_{j=0}^{m-1} B_j v \, F_j u \, dS$$

holds for each $u, v \in C^\infty(\bar{\Omega})$.

<u>Definition 3.1.</u> Let $p \ge 2$, $f_\alpha \in L^p(\Omega)$ for $|\alpha| \le m$, $u_0 \in W_p^m(\Omega)$, $g_j \in L^p(\partial\Omega)$ for $r \le j \le m-1$.

A function $u \in W_p^m(\Omega)$ is said to be a solution of the variational problem

$(3.1)_a$ $$u - u_0 \in V_p,$$

$(3.1)_b$ $$B(u,v) = \int_\Omega \sum_{|\alpha| \le m} f_\alpha D^\alpha v \, dx + \int_{\partial\Omega} \sum_{j=r}^{m-1} g_j B_j v \, dS,$$

if $(3.1)_a$ is satisfied and $(3.1)_b$ holds for each $v \in V_{p'}$.

<u>Lemma 3.1.</u> Let $q > 2$. Then there exists $c_q > 0$ so that

for each $p \in \langle 2,q \rangle$, $f_\alpha \in L^p(\Omega)$, $|\alpha| \leq m$, there is a unique solution $u_p \in W_p^m(\Omega)$ of the variational problem

$$(3.2) \qquad \begin{cases} u \in V_p, \\ \widetilde{B}(u,v) = \int_\Omega \sum_{|\alpha| \leq m} f_\alpha D^\alpha v \, dx, \end{cases}$$

and

$$(3.3) \qquad \|u_p\|_{m,p} \leq c_q^{1-2/p} \Big(\sum_{|\alpha| \leq m} \|f_\alpha\|_{o,p}^p \Big)^{1/p} .$$

Proof. Let $p \in \langle 2,q \rangle$ and $f_\alpha \in L^p(\Omega)$. For $|\alpha| \leq m$ let $\{f_\alpha^{(n)}\} \subset \mathcal{D}(\Omega)$ be such a sequence that $f_\alpha^{(n)} \longrightarrow f_\alpha$ in $L^p(\Omega)$. The bilinear form B is coercive in V and therefore the operators $B_o, \dots, B_{r-1}, F_r, \dots, F_{m-1}$ cover $\widetilde{A}$ on $\partial\Omega$ (see, e.g.[4]; this holds for more wide class of bilinear forms and boundary operators and a proof different of the one in [4] can be given (to appear)). For each $n \in N$ let $u^{(n)} \in C^\infty(\overline{\Omega})$ be the unique solution of the classical b.v.p.

$$\widetilde{A}\, u^{(n)} = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} f_\alpha^{(n)} \quad \text{on } \Omega,$$

$$B_j\, u^{(n)} = 0 \quad \text{on } \partial\Omega \quad , \quad 0 \leq j \leq r-1 ,$$

$$F_j\, u^{(n)} = 0 \quad \text{on } \partial\Omega \quad , \quad r \leq j \leq m-1 .$$

Then $u^{(n)}$ is a solution of (3.2), where $f_\alpha^{(n)}$ is written in place of f_α. As A is formally selfadjoint, we have

$$\int_\Omega v \widetilde{A} u \, dx = \int_\Omega u \widetilde{A} v \, dx + \int_{\partial\Omega} \sum_{j=0}^{m-1} (B_j u F_j v - B_j v F_j u) dS,$$
$$u, v \in C^\infty(\overline{\Omega}) .$$

By Theorem 2.1 and Remark 2.1 there is $c_p > 0$ such that

$$\|u^{(n)}\|_{m,p} \le c_p \sup_{\substack{v\in V\\ \|v\|_{m,p'}=1}} \int_\Omega v\tilde{A}u\,dx \le$$
$$\le c_p \Big(\sum_{|\alpha|\le m} \|f_\alpha^{(n)}\|_{0,p}^p \Big)^{1/p} .$$

Thus $\{u^{(n)}\}$ is a Cauchy sequence in $W_p^m(\Omega)$. Denote $u_p = \lim u^{(n)}$ in $W_p^m(\Omega)$. Then u_p is the unique solution of (3.2) and

$$\|u_p\|_{m,p} \le c_p \Big(\sum_{|\alpha|\le m} \|f_\alpha\|_{0,p}^p \Big)^{1/p} .$$

The constant c_2 can be taken equal to 1. Now, let us interpolate between 2 and q according to Riesz-Thorin's theorem (see, e.g.[8]) and (3.3) follows.

<u>Lemma 3.2.</u> Let $q > 2$. Then there exists $c_q > 0$ so that for each $p \in \langle 2,q \rangle$, $g_j \in L^p(\partial\Omega)$, $r \le j \le m-1$, there is a unique solution $u_p \in W_p^m(\Omega)$ of the variational problem

$u \in V_p$,

$$\tilde{B}(u,v) = \int_{\partial\Omega} \sum_{j=r}^{m-1} g_j\, B_j\, v\, d\,S$$

and

$$\|u_p\|_{m,p} \le c_q \Big(\sum_{j=r}^{m-1} \|g_j\|_{o,p}^p \Big)^{1/p} .$$

The proof is similar to the one of Lemma 3.1 and is omitted. Note only that there is used the fact that $L^p(\partial\Omega) \subset W_p^{-s-1/p}(\partial\Omega)$ with continuous injection for $s \ge 1$, integer.

<u>Corollary 3.1.</u> Let $q > 2$. Then there exists $c_q > 0$ so that for each $p \in \langle 2,q \rangle$, $f_\alpha \in L^p(\Omega)$, $|\alpha| \le m$, and $g_j \in L^p(\partial\Omega)$, $r \le j \le m-1$, there is a unique solution $u_p \in W_p^m(\Omega)$

of the variational problem

$$u \in V_p \;,$$
$$\tilde{B}(u,v) = \int_\Omega \sum_{|\alpha|\leq m} f_\alpha D^\alpha v \, dx + \int_{\partial\Omega} \sum_{j=r}^{m-1} g_j B_j v \, dS$$

and

$$\|u_p\|_{m,p} \leq c_q^{1-2/p} \Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{0,p}^p \Big)^{1/p} + c_q \Big(\sum_{j=r}^{m-1} \|g_j\|_{0,p}^p \Big)^{1/p} .$$

<u>Remark 3.1.</u> Using the same proof method, one can prove the following assertion: If $a_{\alpha\beta} \in C^\infty(\bar{\Omega})$ and A is formally selfadjoint and V-elliptic (i.e. $A = A'$ and there is $c > 0$ such that $B(v,v) \geq c \|v\|^2_{m,2}$ for $v \in V$), then for each $q > 2$ there is $c_q > 0$ such that for each $p \in \langle 2,q \rangle$, $f_\alpha \in L^p(\Omega)$, $|\alpha| \leq m$, and $g_j \in L^p(\partial\Omega)$, $r \leq j \leq m-1$, there is a unique solution $u_p \in W_p^m(\Omega)$ of $(3.1)_a$, $(3.1)_b$ with $u_o = 0$ and

$$\|u_p\|_{m,p} \leq c_q \Big[\Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{0,p}^p \Big)^{1/p} + \Big(\sum_{j=r}^{m-1} \|g_j\|_{0,p}^p \Big)^{1/p} \Big] .$$

<u>Theorem 3.1.</u> Let $a_{\alpha\beta} \in L^\infty(\Omega)$ and $a_{\alpha\beta} = a_{\beta\alpha}$ for $|\alpha|, |\beta| \leq m$ and

$$(3.4) \quad c_1 |\xi|^2 \leq \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x) \xi_\alpha \xi_\beta \leq c_2 |\xi|^2$$

uniformly in Ω for some $0 < c_1 < c_2$. Then there exists $P > 2$ so that for each $p \in \langle 2,P)$ there is $c(p) > 0$ such that for each $f_\alpha \in L^p(\Omega)$, $|\alpha| \leq m$, $u_o \in W_p^m(\Omega)$, and $g_j \in L^p(\partial\Omega)$, $r \leq j \leq m-1$, there is a unique solution $u_p \in W_s^m(\Omega)$ of $(3.1)_a$, $(3.1)_b$ and

$$(3.5) \quad \|u_p\|_{m,p} \leq c(p) \Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{0,p} +$$

$$+\sum_{j=0}^{r-1} \| B_j u_0 \|_{m-m_j-1/p,p} + \sum_{j=r}^{m-1} \| g_j \|_{0,p}) .$$

Proof. At first suppose that $a_{\alpha\beta} \in C^{\infty}(\overline{\Omega})$ and $u_0 = 0$. Let $q > 2$ be arbitrary, $p \in \langle 2, q \rangle$, $f_\alpha \in L^p(\Omega)$, $g_j \in L^p(\partial\Omega)$. Then by Remark 3.1 there is a unique solution u of $(3.1)_a$, $(3.1)_b$, which belongs to $W_p^m(\Omega)$. The equality

$$\tilde{B}(u,v) = \int_\Omega \sum_{|\alpha| \le m} (\sum_{|\beta| \le m} (\delta_{\alpha\beta} - c_2^{-1} a_{\alpha\beta}) D^\beta u) D^\alpha v \, dx$$

$$+ c_2^{-1} \int_\Omega \sum_{|\alpha| \le m} f_\alpha D^\alpha v \, dx + c_2^{-1} \int_{\partial\Omega} \sum_{j=r}^{m-1} g_j B_j v \, dS$$

is satisfied for each $v \in V_{p'}$. If c_3 is the number of all α, for which $|\alpha| \le m$, then

$$(\int_\Omega \sum_{|\alpha| \le m} | \sum_{|\beta| \le m} (\delta_{\alpha\beta} - c_2^{-1} a_{\alpha\beta}) D^\beta u |^p dx)^{1/p} =$$

$$= \sup_{\sum_{|\alpha| \le m} \| h_\alpha \|_{0,p'}^{p'} = 1} \int_\Omega \sum_{|\alpha|, |\beta| \le m} (\delta_{\alpha\beta} - c_2^{-1} a_{\alpha\beta}) h_\alpha D^\beta u \, dx \le$$

$$\le \sup \int_\Omega (\sum_{|\alpha|, |\beta| \le m} (\delta_{\alpha\beta} - c_2^{-1} a_{\alpha\beta}) h_\alpha h_\beta)^{1/2} (\sum_{|\alpha|, |\beta| \le m} (\delta_{\alpha\beta} - c_2^{-1} a_{\alpha\beta}) D^\alpha u \, D^\beta u)^{1/2} dx \le$$

$$\le (1 - c_1 c_2^{-1}) \sup \int_\Omega (\sum_{|\alpha| \le m} h_\alpha^2)^{1/p} \cdot (\sum_{|\alpha| \le m} (D^\alpha u)^2)^{1/2} dx \le$$

$$\le (1 - c_1 c_2^{-1}) \sup (\int_\Omega (\sum_{|\alpha| \le m} h_\alpha^2)^{p'/2} dx)^{1/p'} \cdot$$

$$\cdot (\int_\Omega (\sum_{|\alpha| \le m} (D^\alpha u)^2)^{p/2} dx)^{1/p} \le$$

$$\le (1 - c_1 c_2^{-1}) c_3^{\frac{1}{2}(1 - \frac{2}{p})} \| u \|_{m,p} .$$

By Corollary 3.1 there is $c_q > 0$ such that

$$\| u \|_{m,p} \le c_q^{1-2/p} c_2^{-1} \sum_{|\alpha| \le m} \| f_\alpha \|_{0,p} + c_q c_2^{-1} \sum_{j=r}^{m-1} \| g_j \|_{0,p}$$

$$+ (1 - c_1 c_2^{-1})(c_q c_3^{1/2})^{1-2/p} \| u \|_{m,p} .$$

Therefore for $p \in \langle 2,q \rangle$ satisfying

$$(1 - c_1 c_2^{-1})(c_q c_3^{1/2})^{1-2/p} < 1 ,$$

which is equivalent to

$$(3.6) \qquad p < P(q) = \frac{\log c_q^2 c_3}{\log [c_q c_3^{1/2}(1 - c_1 c_2^{-1})]} ,$$

we have

$$(3.7) \quad \| u \|_{m,p} \leq c(p,q)\Big(\sum_{|\alpha| \leq m} \| f_\alpha \|_{0,p} + \sum_{j=r}^{m-1} \| g_j \|_{0,p} \Big) ,$$

where $c(p,q) > 0$ does not depend on f_α and g_j.

Suppose now that $a_{\alpha\beta} \in L^\infty(\Omega)$. For $|\alpha|, |\beta| \leq m$ let $\{ a_{\alpha\beta}^{(n)} \} \subset C^\infty(\Omega)$ be such a sequence that (3.4) holds with $a_{\alpha\beta}^{(n)}$ in place of $a_{\alpha\beta}$ and $a_{\alpha\beta}^{(n)} \longrightarrow a_{\alpha\beta}$ in measure on Ω. Let $\{ u^{(n)} \} \subset W_p^m(\Omega)$ be the sequence of corresponding solutions. By (3.7) we can suppose without loss of generality that $\{ u^{(n)} \}$ is weakly convergent in $W_p^m(\Omega)$ to some function $\tilde{u} \in W_p^m(\Omega)$, whenever p satisfies (3.6). Then $\tilde{u}$ is the unique solution of $(3.1)_a$, $(3.1)_b$ and (3.7) holds for $\tilde{u}$, as well.

Finally, let $0 \neq u_0 \in W_p^m(\Omega)$ and suppose that p satisfies (3.6). Further, suppose (without loss of generality - see, e.g. [7]) that

$$(3.8) \quad \| u_0 \|_{m,p} \leq c(p) \sum_{j=0}^{r-1} \| B_j u_0 \|_{m-m_j-1/p,\,p} ,$$

where $c(p)$ depends only on p and Ω. By the above considerations there is a unique solution $w_p \in W_p^m(\Omega)$ of $(3.1)_a$,

$(3.1)_b$ with homogeneous boundary condition $(3.1)_a$ and with $f_\alpha - \sum_{|\beta|\leq m} a_{\alpha\beta} D^\beta u_o$ in place of f_α in $(3.1)_b$ and

$$(3.9)\quad \|w_p\|_{m,p} \leq c(p,q)\Big(\sum_{|\alpha|\leq m} \|f_\alpha\|_{0,p} + \\ + c_2\, c(p) \sum_{j=0}^{r-1} \|B_j u_0\|_{m-m_j-1/p,p} + \\ + \sum_{j=r}^{m-1} \|g_j\|_{0,p}\,.$$

The function $u_p = w_p + u_o$ is the unique solution of $(3.1)_a$, $(3.1)_b$ and (3.5) follows from (3.8) and (3.9). Obviously, it suffices to set

$$P = \sup_{q>2} \min\,(q, P(q)).$$

The theorem is proved.

Remark 3.2. The condition (3.4) can be weaker for some special problems. For instance, if $\{B_j\}_{j=o}^{r-1}$ is a Dirichlet set of order r on $\partial\Omega$, then it suffices to suppose that $a_{\alpha\beta} = a_{\beta\alpha}$ only for $r \leq |\alpha|, |\beta| \leq m$ and that

$$c_1 |\xi|^2 \leq \sum_{r\leq|\alpha|,|\beta|\leq m} a_{\alpha\beta}(x)\, \xi_\alpha\, \xi_\beta \leq c_2 |\xi|^2$$

uniformly on Ω .

Remark 3.3. The conditions on $\partial\Omega$ and coefficients of $\{B_j\}_{j=o}^{m-1}$ need not be so strong as they are supposed in Theorem 3.1. Analysing methods of proofs, we see that validity of used a priori estimates and an existence of classical solutions of the considered classical boundary value problems are sufficient. The weakened conditions on the existence of classical solutions are described in [1] and one can verify that proofs of a priori estimates in [6] (which are based on [1]) remain valid under those conditions, too.

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